Econ 101A – Problem Set 6
Due on Tu. May 2nd by 5pm. No late Problem Sets accepted, sorry!

This Problem set tests the knowledge that you accumulated mainly in lectures 20 to 24. The problem set is focused on dynamic games and general equilibrium. General rules for problem sets: show your work, write down the steps that you use to get a solution (no credit for right solutions without explanation), write legibly. If you cannot solve a problem fully, write down a partial solution. We give partial credit for partial solutions that are correct. Do not forget to write your name on the problem set!

Problem 1. Dynamic Games (48 points) Two companies produce the same good. In the first period, firm1 seels its product as a monopolist on the West Coast. In the second period, firm 1 competes with firm 2 on the East Coast as a Cournot duopolist. There is no discounting between the two periods. Firm 1 produces quantity $x_W \leq c/\alpha$ at the West at cost cx_W. On the East Coast, and that's what makes this problem interesting, firm 1 produces quantity x_E at cost $(c - \alpha x_W)\, x_E$, where $0 < \alpha < c < 1/2$. The parameter α captures a form of learning by doing. The more firm 1 produces on the West Coast, the lower the marginal costs are going to be on the East Coast. As for firm 2, it produces in the East market with cost cx_2. The inverse demand functions are $p_W(x_W) = 1 - x_W$ and $p_E(x_E, x_2) = 1 - x_E - x_2$. Each firm maximizes profit. In particular, firm 1 maximizes the total profits from its West and East coast operations.

1. Consider first the case of **simultaneous choice**. Assume that firm 2 does not observe x_W before making its production decision. This means that, although formally firm 1 chooses output x_W first, that you should analyze the game as a symultaneous game between firm 1 and firm 2. Use Nash Equilibrium. Write down the profit function that firm 1 maximizes (careful here) and the profit function that firm 2 maximizes (5 points)
2. Write down the first order conditions of firm 1 with respect to x_W and x_E, and the first order condition of firm 2 with respect to x_2. Solve for x_W^*, x_E^*, and x_2^*. (4 points)
3. Check the second order conditions for firm 1 and for firm 2. (3 points)
4. What is the comparative statics of x_W^* and x_E^* with respect to α? Does it make sense? How about the comparative statics of x_2^* with respect to α? (4 points)
5. Compute the profits of firm 2 in equilibrium. How do they vary as α varies? (compute the comparative statics) Why are firm 2's profits affected by α even though the parameter α does not directly affect the costs of firm 2? (5 points)
6. Now consider the case of **sequential choice**. Assume that firm 2 observes x_W before making its production decision x_2. This means that you should analyze the game as a dynamic game between firm 1 and firm 2, and use the concept of subgame-perfect equilibrium. Remember, we start from the last period. Write down the profit functions that firm 1 and firm 2 maximize on the East Coast (4 points)
7. Write down the first order conditions of firm 1 with respect to x_E, and the first order condition of firm 2 with respect to x_2. Solve for x_E^* and x_2^* as a function of x_W^* (4 points)
8. Compute the comparative statics of x_E^* and x_2^* with respect to x_W^*. Do these results make sense? (3 points)
9. Compute the profits of firm 1 on the East Coast as a function of x_W^*. (2 points)
10. Using the answer to point 9, write down the maximization problem of firm 1 in the first period, that it, when it decides the production on the West Coast. (3 points)
11. Write down the first order conditions of firm 1 with respect to x_W. Solve for x_W^* and then, using the solution for x_W^*, find the solution for x_E^* and x_2^*. (5 points)

12. Compare the solutions for x_W^* under simultaneous and under sequential choice. What can you conclude? Under which conditions the firm does more preemption, that is, produces more on the West Coast in order to reduce the production in equilibrium of firm 2? (6 points)

Problem 2. General Equilibrium (32 points) Consider the case of pure exchange with two consumers. Both consumers have Cobb-Douglas preferences, but with different parameters. Consumer 1 has utility function $u(x_1^1, x_2^1) = (x_1^1)^\alpha (x_2^1)^{1-\alpha}$. Consumer 2 has utility function $u(x_1^2, x_2^2) = (x_1^2)^\beta (x_2^2)^{1-\beta}$. The endowment of good j owned by consumer i is ω_j^i. The price of good 1 is p_1, while the price of good 2 is normalized to 1 without loss of generality.

1. Only for point 1, assume $\omega_1^1 = 1, \omega_1^2 = 3, \omega_2^1 = 3, \omega_2^2 = 1$. (that is, total endowment of each good is 4). Assume further $\alpha = 1/2$, $\beta = 1/2$. Draw the Pareto set and the contract curve for this economy in an Edgeworth box. (you do not need to give the exact solutions, only a graphical representation) What is the set of points that could be the outcome under barter in this economy? (5 points)

2. For each consumer, compute the utility maximization problem. Solve for x_j^{i*} for $j = 1, 2$ and $i = 1, 2$ as a function of the price p_1 and of the endowments. [This problem to be solved with closed eyes!] (5 points)

3. Assume again $\omega_1^1 = 1, \omega_1^2 = 3, \omega_2^1 = 3, \omega_2^2 = 1$ and $\alpha = 1/2$, $\beta = 1/2$. Do a qualitative plot of the offer curve for consumer 1. [Trick to do this is to compute the values of x_1^{1*} and x_2^{1*} as you increase p_1 from low levels – notice though that for $p = 0$ there is infinite demand of x_1] What happens to the consumption of good 1 and good 2 as the price p_1 increases? Plot also the offer curve of consumer 2. Graphically, find the intersection, the general equilibrium point. (7 points)

4. We now solve analytically for the general equilibrium. Require that the total sum of the demands for good 1 equals the total sum of the endowments, that is, that $x_1^{1*} + x_1^{2*} = \omega_1^1 + \omega_1^2$. Solve for the general equilibrium price p_1^*. (6 points)

5. What is the comparative statics of p_1^* with respect to the endowment of good 1, that is, with respect to ω_1^i for $i = 1, 2$? What about with respect to the endowment of the other good? Does this make sense? What is the comparative statics of p_1^* with respect to the taste for good 1, that is, with respect to α and β? Does this make sense? (4 points)

6. Now require the same general equilibrium condition in market 2. Solve for p_1^* again, and check that this solution is the same as the one you found in the point above. In other words, you found a property that is called Walras'Law.In an economy with n markets, if $n - 1$ markets are in equilibrium, the nth market will be in equilibrium as well. (5 points)

Problem 3. Moral Hazard (46 points, due to Botond Koszegi). We analyze here a principal-agent problem with hidden action (moral hazard). The principal is hiring an agent. The agent can put high effort e_H or low effort e_L. If the agent puts high effort e_H, the output is $y_H = 18$ with probability 3/4 and $y_L = 1$ with probability 1/4. If the agent puts low effort e_L, the output is $y_H = 18$ with probability 1/4 and $y_L = 1$ with probability 3/4. The principal decides the pay of the agent w. The utility of the agent is $\sqrt{w} - c(e)$, where $c(e_H) = .1$ and $c(e_L) = 0$. The reservation utility of the agent is .1. The principal maximizes expected profits.

1. Assume first that the principal can observe the effort of the agent (that is, there is no hidden action). We want to determine the optimal contract. In this case, the principal pays a wage $w(e)$ that can depend on the effort. The way you solve a problem like this with a discrete number of effort levels is

to study first the case in which the principal wants to implement high effort e_H. In this case, the wage will be w if the agent chooses e_H and 0 otherwise. Solve the problem

$$\max_w \frac{3}{4}18 + \frac{1}{4}1 - w$$
$$s.t. \sqrt{w} - c(e_H) \geq .1$$

Argue that the constraint is satisfied with equality and solve for w^*. Compute the expected profit in this case $E\pi_H$. (5 points)

2. We are still in the case of no hidden action. Assume that the principal wants to implement low effort e_L. Similarly to above, set up the maximization problem and solve for w^*. Compute the expected profit in this case $E\pi_L$ and compare with $E\pi_H$. Which profit level is higher? The higher one is the contract chosen by the principal and hence the action implemented. (5 points)

3. Now consider the case with hidden action. The wages can only be a function of the outcomes: w_H when $y = y_H$ and w_L when $y = y_L$. We study this case in two steps. Assume first that the principle wants to implement e_H. We study the optimal behavior of the agent after signing the contract. Write the inequality that indicates under what condition the agent prefers action e_H to action e_L. (This is a function of w_H and w_L and goes under the name of incentive compatibility constraint). (5 points)

4. Now consider the condition under which the agent prefers the contract offered to the reservation utility .1. (This is the individual rationality constraint) (4 points)

5. Argue that the two inequalities you just derived will be satisfied with equality. Solve the two equations to derive w_L^* and w_H^*. Compute the expected profits for the principal from implementing the high action under hidden action: $E\pi_H^{HA}$. (5 points)

6. Now, assume that the principal wants the agent to take the low action e_L. In this case, you do not need to worry that the agent will deviate to the high action, since that will take more effort. The principal will pay a flat wage w. Write the individual rationality constraint for the agent if he takes the action e_L and the pay is w. Set this constraint to equality (Why?) and derive w^*. Compute the implied profits for the principal from implementing the low action under hidden action: $E\pi_L^{HA}$.(5 points)

7. Compare the profits in point 5 (profits from high effort) and in point 6 (profits from low effort). Under hidden action, what contract does the principal choose to implement, the one that guarantees high effort or the one that guarantees low effort? (4 points)

8. Compare the profits for the principal under perfect observability and under hidden action. Why are they different despite the fact that the action chosen by the agent in equilibrium will be the same? (6 points)

9. There is a monitoring system that allows the principal to perfectly observe the actions (and hence to implement the perfect observability contract). How much would the principal be willing to pay for it? (3 points)

10. Compare the utility of the agent under perfect observability and under hidden action. (4 points)