Econ 101A – Solution to Midterm 1

Problem 1. Utility maximization. (75 points) In this exercise, we consider a standard utility maximization problem with an unusual (for us) income. The utility function is

$$u(x,y) = \alpha \log(x) + (1-\alpha)\log(y).$$

This function is well-defined for $x > 0$ and for $y > 0$. From now on, assume $x > 0$ and $y > 0$ unless otherwise stated. The price of good x is p_x and the price of good y is p_y. The unusual part is that consumers' income is given not by a monetary budget M, but by endowments of the goods. To be more precise, the consumer starts off owning $\omega_x > 0$ units of good x and $\omega_y > 0$ units of good y. (These numbers are given, treat them as parameters) The consumer can sell the goods so the initial allocation works as a source of wealth. Given this, the income of the consumer is $p_x\omega_x + p_y\omega_y$.

1. Write down the budget constraint. Notice that the budget constraint is standard other than for the income expression, which is given above (5 points)
2. Just for this point, we pick some specific parameter values. Plot the budget lines assuming $\omega_x = 1$, $\omega_y = 1$. Plot first for the case $p_x = 1$, $p_y = 1$, and then plot the budget line for $p_x = 2$, $p_y = 1$. How does the budget line shift? In a new graph, plot the budget constraint for the same two combinations of prices, but assuming $M = 2$. How does the shift in budget line in the case with endowments ω_x and ω_y compare to the standard case with income M? Provide intuition. (10 points)
3. Let's go back to the general case. The consumer maximizes utility subject to the budget constraint with endowments as in point (1). Write down the maximization problem of the consumer with respect to x and y. Explain *briefly* why the budget constraint is satisfied with equality.(5 points)
4. Write down the Lagrangean function. (5 points)
5. Write down the first order conditions for this problem with respect to x, y, and λ. (5 points)
6. Solve explicitly for x^* and y^* as a function of p_x, p_y, ω_x, and ω_y. (5 points)
7. Use the expression for x^* that you obtained in point 6. Differentiate it with respect to p_x, that is, compute $\partial x^*/\partial p_x$. Is this good a Giffen good? (5 points)
8. We are now interested in the sign of $\partial x^*/\partial p_y$. That is, we would like to know if how the demand for one good depends on the price of the other good. Based on this, are these goods (gross) substitutes or complements? (5 points)
9. Consider now how the demand for x^* varies as the endowments ω_x and ω_y vary. That is, compute $\partial x^*/\partial \omega_x$ and $\partial x^*/\partial \omega_y$. Provide an interpretation of these results. (5 points)
10. So far you derived the optimal demands x^* and y^*. Now, consider the demands net of the endowments, that is $z_x^* = x^* - \omega_x$ and $z_y^* = y^* - \omega_y$. These net demands represent the *net demand* for a good. If the net demand is positive, it means that the consumer buys additional quantities relative to the endowment. If the net demand is negative, it indicates that the consumer sells (at least partially) the endowment. Derive expressions for z_x^* and z_y^*. (5 points)
11. Compute how the net demand for good x, that is, z_x^*, varies as the endowment for x varies, that is, compute $\partial z_x^*/\partial \omega_x$. Compare to your answer in point (9) regarding $\partial x^*/\partial \omega_x$ and interpret the difference. (10 points)
12. Suppose now that you have to solve the same problem as above, that is, same budget constraint as in point (1), but the utility function is $v(x,y) = x^a y^{1-\alpha}$. Can you derive the solution for this problem without redoing all the algebra? Explain. (10 point)

Solution to Problem 1.

1. The budget constraint is
$$p_x x + p_y y \leq p_x \omega_x + p_y \omega_y$$

2. The difference from the standard case is that the budget line pivots around the endowment (ω_x, ω_y). In the standard case, it pivots around the axis intercept of the good that does not change price (in this case, good y).

3. The maximization problem is
$$\begin{aligned} & \max_{x,y} \alpha \log(x) + (1-\alpha)\log(y) \\ \text{s.t.} \quad & p_x x + p_y y \leq p_x \omega_x + p_y \omega_y, \\ & x \geq 0 \text{ and } y \geq 0 \end{aligned}$$

   We can write down the budget constraint with equality because the utility function is strictly increasing both in x and y. (Now follows the long explanation, you were not required to give this.) With this utility function, $\partial u(x,y)/\partial x > 0$, and we can by symmetry show $\partial u(x,y)/\partial y > 0$. Given that utility is strictly increasing in both goods, the consumer will never choose a point on the interior of the budget set, i.e., a point $(\hat{x}, \hat{y})$ such that $p_x \hat{x} + p_y \hat{y} < p_x \omega_x + p_y \omega_y$. The reason is that the consumer could choose a point $(\hat{x}', \hat{y}')$ with $\hat{x}' > \hat{x}$ and $\hat{y}' > \hat{y}$ that still satisfies the budget constraint, i.e., such that $p_x \hat{x}' + p_y \hat{y}' \leq p_x \omega_x + p_y \omega_y$. (just pick $(\hat{x}', \hat{y}')$ sufficiently close to $(\hat{x}, \hat{y})$). But, given the monotonicity of u, the bundle $(\hat{x}', \hat{y}')$ provides a higher utility than the bundle $(\hat{x}, \hat{y})$. Therefore the consumer in the optimum will never choose a bundle $(\hat{x}, \hat{y})$ such that $p_x \hat{x} + p_y \hat{y} < p_x \omega_x + p_y \omega_y$. We can therefore limit ourselves to the points with $p_x x + p_y y = p_x \omega_x + p_y \omega_y$.

4. Lagrangean is $L(x, y, \lambda) = \alpha \log(x) + (1-\alpha)\log(y) - \lambda(p_x x + p_y y - p_x \omega_x - p_y \omega_y)$.

5. First order conditions:
$$\begin{aligned} \frac{\partial L}{\partial x} &= \frac{\alpha}{x} - \lambda^* p_x = 0 \\ \frac{\partial L}{\partial y} &= \frac{(1-\alpha)}{y} - \lambda^* p_y = 0 \\ \frac{\partial L}{\partial \lambda} &= -(p_x x + p_y y - p_x \omega_x - p_y \omega_y) = 0 \end{aligned}$$

6. Using the first two first order conditions, we find
$$\frac{\alpha}{1-\alpha}\frac{y^*}{x^*} = \frac{p_x}{p_y}$$
   or
$$y^* = ((1-\alpha)/\alpha)(p_x/p_y)\, x^*. \tag{1}$$
   We substitute this into the budget constraint to get $p_x x^* + p_y ((1-\alpha)/\alpha)(p_x/p_y)\, x^* = M$ or $p_x x^* (1 + (1-\alpha)/\alpha) = p_x \omega_x + p_y \omega_y$ or
$$x^* = \alpha \frac{p_x \omega_x + p_y \omega_y}{p_x} = \alpha \left[\omega_x + \frac{p_y}{p_x}\omega_y\right]. \tag{2}$$
   We substitute (2) into (1) to get
$$y^* = ((1-\alpha)/\alpha)(p_x/p_y)\, \alpha \frac{p_x \omega_x + p_y \omega_y}{p_x} = (1-\alpha)\frac{p_x \omega_x + p_y \omega_y}{p_y} = (1-\alpha)\left[\omega_y + \frac{p_x}{p_y}\omega_x\right]. \tag{3}$$

7. The derivative of x^* with respect to price p_x is
$$\frac{\partial x^*}{\partial p_x} = -\alpha \frac{\omega_y p_y}{p_x^2} < 0.$$
   This good is a not a Giffen good, since the demand for good x is decreasing in its price.

8. The derivative of x^* with respect to price p_y is

$$\frac{\partial x^*}{\partial p_y} = \alpha \frac{\omega_y}{p_x} > 0.$$

The demand for good x increases in the price of good y. Hence, the two goods are (gross) substitutes.

9. The derivatives of x^* with respect to the endowments are

$$\frac{\partial x^*}{\partial \omega_x} = \alpha > 0 \text{ and } \frac{\partial x^*}{\partial \omega_y} = \alpha \frac{p_y}{p_x} > 0.$$

An increase in the endowments makes the agent richer. Since these preferences imply that the goods are normal good, higher endowments of either good lead to more demand.

10. Using the expressions above for x^* and y^*, we obtain

$$\begin{aligned} z_x^* &= x^* - \omega_x = \alpha \left[\omega_x + \frac{p_y}{p_x}\omega_y\right] - \omega_x = \alpha \frac{p_y}{p_x}\omega_y - (1-\alpha)\,\omega_x \\ z_y^* &= x^* - \omega_y = (1-\alpha)\left[\omega_y + \frac{p_x}{p_y}\omega_x\right] - \omega_y = (1-\alpha)\,\frac{p_x}{p_y}\omega_x - \alpha\omega_y \end{aligned}$$

11. The net demand for good x, z_x^*, decreases as the endowment of good x, ω_x, increases. This is unlike what we showed above, in that the demand x^* goes up in ω_x. What is happening is that as the endowment of x goes up, the individuals are richer, but they do not spend all of the extra income into good x, deciding to spend some of it into good y. Formally, the demand for x^* goes up by α for an increase in ω_x, which is less than one. That is why the net demand is actually decreasing in the endowment of x: being richer in x leads to some increase in consumption of x, some increase in y, leading to a decrease in net demand for x.

12. Notice that the utility function $v(x,y)$, which is a standard Cobb-Douglas, is a monotonic transformation of the utility function $u(x,y)$, with $v(x,y) = \exp(u(x,y))$. Hence, the two utility functions represent the same preferences, and thus the two utility maximization problems are identical. As such, the solution would be identical.

Short problem. (Utility Maximization with Graphical Solution) (40 points)

1. Consider an individual with budget constraint $2x + y = 10$. That is, price of x is $p_x = 2$, price of y is $p_y = 1$, and income M equals 10. Plot the budget constraint. (5 points)
2. Plot indifferent curves for the case $u(x, y) = x^2 + y^2$. (Remember: an indifference curve is defined by $u(u, y) - \bar{u} = 0$). (5 points)
3. Using the plots of the indifference curves and the plot of the budget constraint, find graphically the utility-maximizing choice, and solve for x^* and y^*. (Remember that you are maximizing utility subject to the budget constraint and subject to $x \geq 0$ and $y \geq 0$) (10 points)
4. Why in the above case it would not be a good idea to solve the problem with a Lagrangean? (5 points)
5. Now, using the same budget constraint $2x + y = 10$, plot indifference curves for the case $u(x, y) = 4x + y$.(5 points)
6. Using the plots of the indifference curves and the plot of the budget constraint, find graphically the utility-maximizing choice, and solve for x^* and y^*. (5 points)
7. What preferences are the ones in points (2) and (5)? What cases do they indicate? (5 points)

Solution to Short Problem.

1. The budget constraint is simply given by the equation $y = 10 - 2x$, it is a straight line crossing the y axis at 10 and the x axis at 5.
2. These indifference curves are circles centered in the origin.
3. Graphically, one can see that the point of tangency between the indifference curves (the circles) and the budget lines are not optimal points, in fact they would minimize utility. The solution in this case it to go to one of the corner solutions with either $x = 0$ or $y = 0$. The solution for $x = 0$ would be $(x^*, y^*) = (0, 10)$, yielding utility $10^2 = 100$. The solution for $y = 0$ would be $(x^*, y^*) = (5, 0)$, yielding utility $5^2 = 25$. Hence, the solution is $(x^*, y^*) = (0, 10)$, as you should be able to see graphically.
4. If you were to solve the Lagrangrean, given that the preferences are not convex, you would be finding a minimum, not a maximum, you would find the tangency point highlighted above.
5. The indifference curves are straight lines with slope -4
6. The utility-maximizing solutions will once again be at the corners, this time for $(x^*, y^*) = (5, 0)$, yielding utility 20. At the other corner (0,10), the utility is only 10.
7. The preferences in point (2) and examples of the CES family with $\rho = 2$. The preferences in point (5) are cases of perfect substitutes, in which 4 units of good y are equivalent to one unit of good x.