Econ 101A – Midterm 1
Tu 20 February 2012.

You have approximately 1 hour and 20 minutes to answer the questions in the midterm. We will collect the exams at 11.00 sharp. Show your work, and good luck!

Problem 1. Consumption and Leisure with Quasi-linear utility. (70 points + 6 extra credit) Each day, Elisabetta has quasi-linear utility over leisure and consumption. In particular, she maximizes $U(l, c; w) = \phi(l) + c + \psi(w)$. As we discussed in class, she can only spend her endowment M and wages earned wh, where $h \leq 24$ are her hours spent working to earn hourly wage w. The price of consumption is p.

We make the standard assumption on the utility for leisure $\phi'(l) > 0$, $\phi''(l) < 0$, as well as $\phi'(0) = 1$. In addition, note that Elisabetta also derives utility from her hourly wage through $\psi(w)$, with $\psi(w)$ differentiable for all $w \geq 0$. You can think of $\psi(w)$ as measuring the indirect pleasure/guilt Elisabetta receives from earning a high wage. If $\psi'(w) < 0$, Elisabetta derives guilt from earning a high wage, and if $\psi'(w) > 0$, she derives pleasure.

1. Derive Elisabetta's budget constraint. (4 points)
2. Write down the maximization problem of the worker with respect to c and l with *all* the relevant constraints. Assume that the budget constraint is satisfied with equality. (2 points)
3. Write down the Lagrangean and derive the first order conditions with respect to c, l, and λ. (4 points)
4. Simplify the problem to a system of two equations in two unknowns, c^* and l^* by solving for λ^*. This system defines the solution when the solution is interior. (4 points)
5. In this interior solution, how much does pride and shame (as captured by $\psi(w)$) matter? Provide economic intuition or mathematical intuition (or both!) for why you get this result. (6 points)
6. Assuming an interior solution, check the second order conditions for a maximum using the bordered Hessian. (5 points)
7. Still assuming an interior solution, use the implicit function theorem to compute (i) dl^*/dw; (ii) dl^*/dM. (Suggestion: Use the uni-variate implicit function theorem) Also, compute directly dc^*/dM (you do not need the implicit function theorem here) For each of these comparative statics, provide *interpretation* (10 points)
8. Which of the responses in the previous point (7) are affected by the quasi-linearity assumption? That is, compare qualitatively to the responses to (7) you would have gotten if the utility function had been $U(l, c; w) = l^\alpha c^{1-\alpha} + \psi(w)$. [You should not need to solve the latter problem from scratch to answer this question] (7 points)
9. Going back to the system in point (4), characterize *when* a corner solution occurs, and what that solution is. (8 points for checking the case $l \geq 0$, *extra credit* of up to 6 more points for doing all corner solutions)
10. (Question with less guidance, worth 20 points) Now assume that the utility function instead is $U(l, c; w) = \phi(l)\psi(w) + c$. All assumptions are as above, with the additional assumption that $\psi(w) > 0$. In this case, the pride/shame term affect the marginal utility of leisure. Solve the maximization problem with this alternative utility function, and compare the interior solutions found here to the ones found above. How much does pride and shame (as captured by $\psi(w)$) matter now? Compute dl^*/dw now and compare to your answer in point (7) above. Provide interpretation. (20 points)

Solution to Problem 1.

1. The budget constraint is that Elisabetta cannot spend more than she earns, and hence

$$pc \leq M + wh.$$

This can be rewritten as

$$\begin{aligned} pc &\leq M + w(24 - l) \\ pc + wl &\leq M + 24w \end{aligned}$$

2. The maximization problem is

$$\begin{aligned} &\max_{l,c} \phi(l) + c + \psi(w) \\ &s.t.\ pc + wl \leq M + 24w \\ &s.t. 0 \leq l \leq 24 \\ &s.t. c \geq 0 \end{aligned}$$

3. The Lagrangean is

$$L\left(c, l, \lambda; p, w, M\right) = \phi(l) + c + \psi(w) - \lambda\left(pc + wl - M - 24w\right)$$

The first order conditions are

$$\frac{\partial}{\partial c} : 1 - \lambda p = 0$$

$$\frac{\partial}{\partial l} : \phi'(l) - \lambda w = 0$$

$$\frac{\partial}{\partial \lambda} : -pc - wl + M + 24w = 0$$

4. We can solve for $\lambda^* = 1/p$ from the first condition above and obtain

$$\begin{aligned} \phi'(l) - \frac{w}{p} &= 0 \\ c^* &= \frac{M + w\left(24 - l^*\right)}{p} \end{aligned}$$

5. The term for pride and shame drops out from the first order conditions and hence does not affect the solution. This is because, while the person does get pride and shame, the pride and shame do *not* interact with the other terms in the utility function. Mathematically, the term $\psi(w)$ is additively separable from the rest of the problem. Intuitively, Elisabetta may be happy or not that the wage is high (or low), but she cannot choose the wage, and whatever the wage is, this term does not affect the optimal choice of c and l..

6. We need to check for s.o.c. the bordered Hessian, which is as follows

$$\begin{aligned} |H_L| &= \left| \begin{bmatrix} 0 & -w & -p \\ -w & \phi''(l) & 0 \\ -p & 0 & 0 \end{bmatrix} \right| \\ &= 0 + 0 - p * (p\phi''(l)) = -\phi''(l)p^2 > 0 \end{aligned}$$

by the assumption of $\phi\left(l\right)$ concave. Since the determinant is positive, the condition is satisfied.

7. To compute the comparative statics, we use the implicit equation

$$\phi'(l) - \frac{w}{p} = 0.$$

Notice that this equation defines l independently of c, so there is no need to consider the two equations together. (This is only true because the utility function is quasi-linear).

(a) Applying the implicit function theorem, we obtain

$$\frac{dl^*}{dw} = -\frac{-1/p}{\phi''(l^*)} < 0$$

by the concavity of $\phi(l)$. Hence, increasing the salary leads to a decrease in leisure. In this case, there is a pure substitution effect, that higher wage makes it more expensive not to be working. The fact that the function is quasi-linear means that there is no income effect of an increase in wage w, which could have reversed this result. [You did not need to get the second part of the intuition for full credit].

(b) We can also compute

$$\frac{dl^*}{dM} = -\frac{0}{\phi''(l^*)} = 0.$$

In this case, an increase in income has no effect on the allocation of leisure. This is again because of the special quasi-linear structure, you can see that there are no income effects here.

(c) Finally, we can compute

$$\frac{dc^*}{dM} = \frac{d}{dM}\left(\frac{M + w(24 - l^*)}{p}\right) = 1 - \frac{w}{p}\frac{dl^*}{dM} = 1.$$

Notice that here we used the result which we found above, $dl^*/dM = 0$. This result implies that each extra dollar earned gets all spent into consumption, not into leisure. This is again because of the special quasi-linear utility function.

8. I commented on this above. To repeat:

(a) In general, dl^*/dw can be negative or positive depending on the income effect, which adds to the substitution effect (which is always negative). But in the problem above there was no income effect

(b) A Cobb-Douglas function would have an income effect, and in particular would have $dl^*/dM > 0$. That is, a higher income leads to more leisure, and less work. The quasi-linearity above implies no income effect.

(c) As a result of the response above, all the extra income is spent on the consumption good. In the Cobb-Douglas case, this would not be true, and consumption wold go up less than one-to-one with income M, because the agent uses the higher income to work less, and hence can afford less consumption good.

9. Remember that the solutions have to satisfy the constraints

$$\begin{aligned} 0 &\leq l \leq 24 \text{ and} \\ c &\geq 0 \end{aligned}$$

(a) For the first constraint, we first check $l \geq 0$. Remember that the first-order condition for l satisfies

$$\phi'(l) - \frac{w}{p} = 0,$$

and that we assumed $\phi'(0) = 1$. Hence, as long as $w/p > 1$, $\phi'(l) > \phi'(0)$ and since $\phi'(l)$ is decreasing in l ($\phi(l)$ is concave), this would imply $l < 0$, violating the constraints. Hence, for w/p, the solution will instead be a corner solution:

$$\begin{aligned} l^* &= 0 \\ c^* &= \frac{M + w24}{p}. \end{aligned}$$

Intuitively, when the wage is very high, Elisabetta wants to work very hard up to the point of getting... negative leisure!

(b) For extra credit, let us now also check the constrains $l \leq 24$. From the first order condition, we have to worry if

$$\phi'(24) - w/p > 0,$$

or $w/p < \phi'(24)$. In this case, Elisabetta would like to take more than 24 hours of leisure because the (real) wage is so low. Hence, we hit the corner solution

$$\begin{aligned} l^* &= 24 \\ c^* &= \frac{M}{p}. \end{aligned}$$

(c) Finally, we need to check $c \geq 0$. We can notice though that, as long as $M \geq 0$, the condition which we checked above that $l \leq 24$ implies that the constraint $c \geq 0$ is also satisfied. Hence, this constraint is redundant, that is, automatically satisfied once the constraint on l is satisfied.

10. The utility function now is $U(l, c; w) = \phi(l)\psi(w) + c$, which leads to the maximization problem

$$\begin{aligned} &\max_{l,c} \phi(l)\psi(w) + c \\ &s.t.\ pc + wl \leq M + 24w \\ &s.t. 0 \leq l \leq 24 \\ &s.t. c \geq 0 \end{aligned}$$

and the Lagrangean

$$L(c, l, \lambda; p, w, M) = \phi(l)\psi(w) + c - \lambda(pc + wl - M - 24w)$$

The first order conditions are

$$\frac{\partial}{\partial c} : 1 - \lambda p = 0$$

$$\frac{\partial}{\partial l} : \phi'(l)\psi(w) - \lambda w = 0$$

$$\frac{\partial}{\partial \lambda} : -pc - wl + M + 24w = 0$$

Like above, we solve for $\lambda^* = 1/p$ and obtain

$$\begin{aligned} \phi'(l)\psi(w) - \frac{w}{p} &= 0 \\ c^* &= \frac{M + w(24 - l^*)}{p} \end{aligned}$$

The pride and shame term now affects the first order condition for leisure l, unlike in the previous case. In particular, the interpretation of $\psi'(w)$ changes: if $\psi'(w) > 0$ then the marginal utility of leisure inreases with w, increasing the optimal level of leisure. Thus, an increasing $\psi(w)$ should be interpreted as shame or stigma from earning a higher wage (and if $\psi'(w) < 0$, pride from earning a higher wage). Let us compute dl^*/dw using the implicit function theorem:

$$\frac{dl^*}{dw} = -\frac{\phi'(l^*)\psi'(w) - 1/p}{\phi''(l^*)\psi(w)}.$$

Notice that the expression dl^*/dw changed. There are now two parts to it. The second part, still captured in the numerator by $-1/p$, is the same substitution effect which we saw before, that a higher wage w increases the shadow price of leisure. But in addition an increase in wage also has another effect through the pride/guilt component, as one can see from $\phi'(l^*)\psi'(w)$. Since $\phi'(l^*)$ is positive, this term is positive if and only if $\psi'(w)$ is positive, which captures the case of stigma from high wage. In this case, a higher wage increases the marginal utility of leisure (by how the utility function is) and leads to more leisure. Hence, this introduces an effect opposite to the substitution effect. The opposite occurs if there is pride, and $\psi'(w)$ is negative. In this case, an increase in w depresses the marginal utility of leisure and leads to a lower optimal leisure.

Problem 2. Preferences. (30 points)

Consider the following preferences defined on $X = R^+$ as follows: $x \succsim y$ if $x \geq y$ or $x \geq M$. [Notice that the preferences here are over just one good, not two as we have often considered, that is, x and y are scalars, not vectors; to say this otherwise, think of x or y as number of apples]

1. (a) Provide an economic interpretation for this preference relation. In particular, how do you interpret M? (8 points)
   (b) Is this preference relation rational? Define and show formally if you can (6 points)
   (c) Find a utility function that represent these preferences, and show that it represents them. Is there only one such function? (10 points)
   (d) Is this preference relation (weakly) monotonic? Is it strongly monotonic? (6 points)

Solution to Problem 2.

1. We can interpret the preferences above as follows. This individual likes more apples to less, but reaches a satiation point at M, beyond which point the individual is indifferent about any higher level of consumption. Notice that, for any x and y both above the satiation level M, both $x \succsim y$ and $y \succsim x$ hold, that is, x and y are indifferent.

2. To prove that these preferences are rational, we need to prove that they are (i) complete and (ii) transitive.

   (a) Proof that they are complete. We need to show that for all x and y, either $x \succsim y$ holds, or $y \succsim x$ holds, or both. Consider first the case in which $x \leq y \leq M$. In this case $y \succsim x$ holds. Similarly, in the case $y \leq x \leq M$, $x \succsim y$ holds. Then consider the case $x \leq M \leq y$. In this case, $y \succsim x$ holds. Similarly, in the case $y \leq M \leq x$, $x \succsim y$ holds. Only the case $M \leq \min(x, y)$ remains, and in this case, both $x \succsim y$ and $y \succsim x$ hold. So the preferences are complete

   (b) Proof that they are transitive. Assume $x \succsim y$ and $y \succsim z$. Then, it must be the case that $x \succsim z$ holds. To prove this, consider that $x \succsim y$ implies either $x \geq y$ or $x \geq M$, or both. If the latter condition (that is, $x \geq M$) holds, then we are done, since that would imply that $x \succsim z$ holds by definition. So we consider the case in which it does not hold, and $x \geq y$ holds. We also know that $y \succsim z$, and hence either $y \geq z$ or $y \geq M$. If the former holds, then $x \geq y \geq z$ implies $x \geq z$ and hence $x \succsim z$ holds by definition. If the former does not hold, then $y \geq M$ holds, and $x \geq y \geq M$ implies $x \geq M$, and hence $x \succsim z$ holds by definition. This completes the proof.

3. A utility function that represent the preferences is

$$u(x) = \begin{cases} x & \text{if} \quad x < M \\ M & \text{if} \quad x \geq M \end{cases}.$$

   We now show that it does indeed represent the preferences, that is, we show that $u(x) \geq u(y)$ if and only if $x \succsim y$. Assume then that $u(x) \geq u(y)$. We now consider four cases; similarly one can prove the other direction.

   (a) If both x and y are below M, this is equivalent to saying that $x \geq y$, and hence $x \succsim y$
   (b) If x is below M, and y is (weakly) above M, then it cannot be that $u(x) \geq u(y)$, so this case is irrelevant
   (c) If y is below M, and x is (weakly) above M, then $u(x) \geq u(y)$ is equivalent to $x \geq y$, and hence $x \succsim y$
   (d) Finally, if x and y are both (weakly) above M, then $u(x) \geq u(y)$ is trivially satisfied since $x \geq M$, and hence $x \succsim y$

   - Notice that the utility function above is not unique as one can repeat the proof, for example, with

$$u(x) = \begin{cases} \alpha x & \text{if} \quad x < M \\ \alpha M & \text{if} \quad x \geq M \end{cases}.$$

     for any $\alpha > 0$. Hence, there is an infinity of utility functions.

4. Proof:

   (a) Weak monotonicity of preference is defined as follows: if $x \geq y$, then $x \succsim y$. This is trivially true given the definition of the preference relation.

   (b) Strong monotonicity is defined (in this setting in which there is only one dimension) as follows: If $x > y$, then $x \succ y$. But the latter definition does not hold. Consider the case in which both x and y are larger than M, but $x > y$. Still, by the definition of preferences, $x \succsim y$ and $y \succsim x$, violating strict monotonicity.