Econ 101A – Midterm 1
Th 1 October 2009.

You have approximately 1 hour and 20 minutes to answer the questions in the midterm. Justin and Mariana will collect the exams at 11.00 sharp. Show your work, and good luck!

Problem 1. Consumption and Leisure Decision: Farmers and Baseball Players. (65 points) Suppose there is an island where there are only 2 types of work available for the islanders: farming and playing baseball. All islanders like playing baseball, while they do not like (per se) farming. The king of the island (randomly) assigns each islander his/her future profession on the day they are born. Islanders are only allowed to work in their assigned profession. This question is about an islander's work decision in their assigned profession when they reach working age.
The utility function for islanders whose profession is farming:

$$u(c, \ell; \alpha, \beta) = c^\alpha \ell^\beta$$

The utility function for islanders whose profession is playing baseball:

$$u(c, \ell, h; \alpha, \beta, \gamma) = c^\alpha \ell^\beta h^\gamma$$

In this problem we will assume that islanders decide how many hours to work each day, where H is the number of hours in a day (i.e. 24) and h is the number of hours that you work each day. All time not spent working is spent on leisure ℓ. Islanders also get utility from consuming all other goods c and the price of these goods is \$1 (i.e. $p_c = 1$). Assume that $\alpha, \beta, \gamma \in (0, 1)$.

1. Provide intuition of what γ is capturing in the utility function of the baseball players. That is, what does this non-standard consumption-leisure utility function capture? (5 points)
2. First notice that the utility function for islanders whose profession is playing baseball is really a utility function of just 2 endogenous variables (c and ℓ). Rewrite this utility function so that the level of utility is a function of only these choice variables and other exogenous parameters. (5 points)
3. Let's assume that the only income for islanders comes exclusively from working and that to be "fair" the king decides to set the wage that an islander receives for an hour of work to be the same regardless of the type of work. Write the budget constraint that an islander faces in their consumption-leisure decision problem. (5 points)
4. Write down the maximization problem, the Lagrangian, and the first order conditions for the consumption-leisure decision for both the farming islander and the baseball playing islander. (10 points)
5. Solve for c^* and ℓ^* (as well as h^*) for the farming islander. Check that the solution satisfies the constraint $0 \leq l^* \leq H$. (10 points)
6. Still for the farming islander, plot the labor supply function, that is, how the hours of work supplied h^* vary with the wage w. (use h^* on the x axis and w on the y axis). What is the particular feature of this function? Relate to the substitution and income effects. (5 points)
7. Check the second order conditions for the candidate solution you found for farming islanders in the previous part. Use the bordered Hessian. (5 points)
8. The solution for the baseball playing islander is $l^* = (\beta/(\alpha + \beta + \gamma)) * H$. (You are not required to solve for this, though if you do solve for it from the first order condition you get 5 extra credit points). Compare this solution to the solution for l^* for the farmers. Discuss the intuition for how the optimal leisure choices differ. How does l^* vary as γ increases? Discuss the intuition. (5 points)
9. Consider now a special type of baseball players, the workaholic ones. These players do not enjoy leisure time l, while they enjoy working h:

   $$u(c, \ell, h; \alpha, \beta, \gamma) = c^\alpha h^\gamma.$$

   Solve for the optimal c^* and l^* in this case. Do not rely on the answer to Question 8. [Hint: Do you need to set up the full Lagrangean to solve for this?] (10 points)

10. The king realizes over time that farmers are unhappy being stuck with their profession. The island needs to have farmers, so the king decides to pay all farmers a transfer of M dollars in addition to their wages. Without solving numerically explain how the king's economic advisor could determine the level of M that each farmer would need to make farmers as happy as (non-workaholic) baseball players. (5 points)

Solution to Problem 1.

1. The parameter γ captures how much the baseball players enjoy working, that is, playing and training. In the standard model, we assume that the agents do not enjoy working per se, but work for the money they earn, in order to buy consumption goods c.

2. The constraint on a fixed number of hours in the day implies $h = H - l$. Substituting this solution for h in the utility function of farmers, we obtain

$$u(c, \ell; \alpha, \beta, \gamma, H) = c^{\alpha} \ell^{\beta} (H - l)^{\gamma}$$

3. The budget constraint is that the amount of money spent on a good cannot be larger than the income:

$$c \leq w (H - l)$$

which can be rewritten as

$$c + wl \leq Hw$$

4. The farmer maximizes

$$\begin{aligned} &\max_{c,l} c^{\alpha} l^{\beta} \\ s.t. c + wl &\leq Hw \\ s.t.\ c &\geq 0 \\ s.t.\ H &\geq l \geq 0 \end{aligned}$$

The Lagrangean is

$$L(c, l) = c^{\alpha} l^{\beta} - \lambda (c + wl - Hw)$$

We are able to write the budget constraint with equality in the Lagrangean because the utility function is strictly increasing with both c and l (we need it only to be strictly increasing for one of the consumption goods).

The first order conditions are

$$\begin{aligned} f.o.c.c &: \alpha c^{\alpha - 1} l^{\beta} - \lambda = 0 \\ f.o.c.l &: \beta c^{\alpha} l^{\beta - 1} - \lambda w = 0 \\ f.o.c.\lambda &: -(c + wl - Hw) = 0 \end{aligned} \tag{1}$$

For the baseball player, the maximization is

$$\begin{aligned} &\max_{c,l} c^{\alpha} l^{\beta} (H - l)^{\gamma} \\ s.t. c + wl &\leq Hw \\ s.t.\ c &\geq 0 \\ s.t.\ H &\geq l \geq 0 \end{aligned}$$

The Lagrangean is

$$L(c, l) = c^{\alpha} l^{\beta} (H - l)^{\gamma} - \lambda (c + wl - Hw)$$

The first order conditions are

$$\begin{aligned} f.o.c.c &: \alpha c^{\alpha-1} l^{\beta} (H-l)^{\gamma} - \lambda = 0 \\ f.o.c.l &: \beta c^{\alpha} l^{\beta-1} (H-l)^{\gamma} - \gamma c^{\alpha} l^{\beta} (H-l)^{\gamma-1} - \lambda w = 0 \\ f.o.c.\lambda &: -(c + wl - Hw) = 0 \end{aligned}$$

5. The solutions for the farmer are obtained from the first two first=order conditions, moving the last term to the right-hand-size and dividing through:

$$\begin{aligned} \frac{\alpha c^{\alpha-1} l^{\beta}}{\beta c^{\alpha} l^{\beta-1}} &= \frac{\lambda}{\lambda w} \text{ or simplifying} \\ \frac{\alpha l}{\beta c} &= \frac{1}{w}, \text{ that is} \\ l &= \frac{\beta c}{\alpha w} \end{aligned}$$

Plugging this into the budget constraint, we obtain

$$\begin{aligned} c + w\frac{\beta c}{\alpha w} &= Hw \text{ or} \\ c^* &= \frac{\alpha}{\alpha+\beta} Hw \text{ and using the expression for } l \\ l^* &= \frac{\beta c}{\alpha w} = \frac{\beta}{\alpha w} \frac{\alpha}{\alpha+\beta} Hw = \frac{\beta}{\alpha+\beta} H \text{ and} \\ h^* &= H - l^* = H - \frac{\beta}{\alpha+\beta} H = \frac{\alpha}{\alpha+\beta} H \end{aligned}$$

6. The demand function for l^* is independent of w, an unusual feature. This is because there are two opposing effects: (i) a substitution effect that leads to a reduction of leisure (and hence an increase in hours worked) when the shadow cost of leisure w goes up; (ii) an income effect that leads to an increase in leisure (and hence a decrease in hours worked) when an increase in the wage w occurs. For a Cobb-Douglas function these two effects cancel each other out.

7. To check the second-order conditions, we compute the Bordered Hessian using the first order conditions at (1)

$$H = \begin{matrix} 0 & -1 & -w \\ -1 & \alpha(\alpha-1) c^{\alpha-2} l^{\beta} & \alpha\beta c^{\alpha-1} l^{\beta-1} \\ -w & \alpha\beta c^{\alpha-1} l^{\beta-1} & \beta(\beta-1) c^{\alpha} l^{\beta-2} \end{matrix}$$

The determinant is

$$\begin{aligned} & 0 - (-1) * \left[-\beta(\beta-1) c^{\alpha} l^{\beta-2} + w\alpha\beta c^{\alpha-1} l^{\beta-1}\right] - w * \left[-\alpha\beta c^{\alpha-1} l^{\beta-1} + w\alpha(\alpha-1) c^{\alpha-2} l^{\beta}\right] \\ = & -\beta(\beta-1) c^{\alpha} l^{\beta-2} + w\alpha\beta c^{\alpha-1} l^{\beta-1} + w\alpha\beta c^{\alpha-1} l^{\beta-1} - w^2 \alpha(\alpha-1) c^{\alpha-2} l^{\beta} \\ = & -\beta(\beta-1) c^{\alpha} l^{\beta-2} + 2w\alpha\beta c^{\alpha-1} l^{\beta-1} - w^2 \alpha(\alpha-1) c^{\alpha-2} l^{\beta} \end{aligned}$$

where all three terms are positive (remember $\alpha < 1$ and $\beta < 1$). Hence the bordered Hessian is positive, as required.

8. We can solve for the optimal amount of leisure time by starting with the three first order equations:

$$\begin{aligned} f.o.c.c &: \alpha c^{\alpha-1} l^{\beta} (H-l)^{\gamma} - \lambda = 0 \\ f.o.c.l &: \beta c^{\alpha} l^{\beta-1} (H-l)^{\gamma} - \gamma c^{\alpha} l^{\beta} (H-l)^{\gamma-1} - \lambda w = 0 \\ f.o.c.\lambda &: -(c + wl - Hw) = 0 \end{aligned}$$

into just one equation. Dividing through the second equation by the first, we obtain

$$\begin{aligned}
\frac{\beta c^{\alpha} l^{\beta-1}\left(H-l\right)^{\gamma}-\gamma c^{\alpha} l^{\beta}\left(H-l\right)^{\gamma-1}}{\alpha c^{\alpha-1} l^{\beta}\left(H-l\right)^{\gamma}} &= w \text{ or} \\
\frac{\beta l^{-1}-\gamma\left(H-l\right)^{-1}}{\alpha c^{-1}} &= w \text{ or} \\
\frac{\frac{\beta(H-l)-\gamma l}{l(H-l)}}{\alpha} c - w &= 0 \text{ or} \\
\frac{\beta l^{-1}-\gamma\left(H-l\right)^{-1}}{\alpha} c - w &= 0
\end{aligned}$$

Plugging in the expression for c in the third equation ($c = (H-l)\,w$), we obtain

$$\begin{aligned}
\frac{\beta l^{-1}-\gamma\left(H-l\right)^{-1}}{\alpha}\left(H-l\right)w - w &= 0 \text{ or} \\
\frac{\beta}{\alpha}\frac{H-l}{l} - \frac{\gamma+\alpha}{\alpha} &= 0
\end{aligned}$$

and hence

$$\begin{aligned}
\beta\left(H-l\right) &= \left(\gamma+\alpha\right) l \\
l^{*} &= \frac{\beta}{\gamma+\alpha+\beta} H
\end{aligned}$$

Next we compare the optimal level of leisure for baseball players and farmers and notice that since $\gamma \in (0,1)$ that $l_f^* > l_b^*$ or that the optimal level of leisure for farmers will always be greater than that of baseball players. Since baseball players derive utility from working in addition to their wage (which is spent on consumption) then baseball players must take into account this loss of utility if they take leisure. Thus the amount of leisure is less than that of farmers, but since the (3 good) Cobb-Douglas utility function is increasing at a decreasing rate we know that baseball players will still take some leisure.

The optimal amount of leisure for baseball players is decreasing in γ: $\frac{dl^*}{d\gamma} = -\frac{HB}{(\alpha+\beta+\gamma)^2} < 0$. This confirms our intuition. The higher is γ, the more baseball players enjoy their work and the more utility they gain from each hour of work. Thus baseball players will work more (and take less leisure).

9. Workaholic baseball players don't derive any pleasure from leisure. Thus, it is easiest not to use the Lagrangian to solve this problem. We know that workaholics will set $l^* = 0$ because they will want to spend all of their time working: $h^* = H$. Since they spend all of their time working they will have income wH to spend on consumption ($c^* = wH$). Another way to see this is if we replace h with $H - l$ in the utility function. Then if we differentiate the utility function with respect to leisure (l) we see that marginal utility of leisure is negative: $\frac{dU}{dl} = -\gamma c^{\alpha}(H-l)^{\gamma-1}$. Thus, we would want to set leisure as low as possible (i.e. zero). If we solve using the usual Lagrangian we have to remember that if we find a "solution" that is a boundary case that we check to see if this is indeed a maximum. For example, it is clear from inspection of the utility function that utility will be zero if either $c = 0$ or $h = 0$.

10. The key to this problem is first recognizing that the king will want to use a lump sum transfer of \$M to equalize the utilities of the farmer and the baseball player. Since we know that farmers and baseball players will optimize, then this implies setting equal the 2 indirect utility functions. Note that it is crucial that we allow the farmers and baseball players decide how many hours to work and how much to spend on consumption good c.

    In order to find the M that will do this, we can first solve for the baseball player's indirect utility by plugging l^*, c^*, h^* into the baseball utility function. Then we solve an expenditure minimization problem for the farmer setting the constraint as $c^{\alpha}\ell^{\beta} - v_b(.) = 0$ (or equal to baseball player indirect utility). Finally, we can use the solutions $c^*(p, w; v_b)$ and $\ell^*(p, w; v_b)$ (i.e. the Hicksian demands) to solve for M by solving: $c^*(p, w; v_b) = wh^*(p, w; v_b) + M$. If we want, we can check that this is the level of M that will equate the indirect utility functions by including this M (call it M') in a new budget constraint for the farmer ($c + wl \leq wH + M'$)and then solving the UMAX problem for the farmer. Finally, note that when we solve for the Hicksian demands for the farmer, it is possible that we may find that the farmer wants to set $l^* > H$. If this is the case then the farmer will set $l^* = H$ and $c^* = M$ and no farmers will be willing to work even with the transfer!

Problem 2. (15 points)

1. Consider a 2-period economy, $t = 0$ and $t = 1$, as considered in class. In each period, the consumer receives income M_t and decides to consume c_t, with $t = 0, 1$. The prices of goods c_t is equal to 1 in all periods. Write down the intertemporal budget constraint that we use in the utility maximization for the consumption-savings case. (Hint: Start from the last period) (5 points)

2. Now let's generalize that to a 3-period economy, $t = 0$, $t = 1$, and $t = 2$. In each period, the consumer receives income M_t and decides to consume c_t, with $t = 0, 1, 2$. The prices of goods c_t is still equal to 1 in all periods. Write down the intertemporal budget constraint that we use in the utility maximization for the consumption-savings case. (Again: Start from the last period) Can you conjecture how the constraint looks like for a t-period economy? (10 points)

Solution to Problem 2.

1. In the last period you can only spend (c_1) as much money as you have left. The amount of money you have left will be equal to the new money from the last period (M_1) along with the money saved (or borrowed) from the first period plus any interest you accrue on the saved/borrowed money: $(M_0 - c_0)(1 + r)$. Thus the 2 period inter-temporal budget constraint becomes:

$$\begin{aligned} c_1 &\leq M_1 + (M_0 - c_0)(1 + r) \Leftrightarrow \\ c_1 + c_0 &\leq M_1 + M_0 + (M_0 - c_0)r \end{aligned}$$

2. We set up the problem the same as in the 2-period case, except that now we have an extra time period:

$$\begin{aligned} c_2 &\leq M_2 + (M_0 - c_0)(1 + r)^2 + (M_1 - c_1)(1 + r) \Leftrightarrow \\ c_0(1 + r)^2 + c_1(1 + r) + c_2 &\leq M_0(1 + r)^2 + M_1(1 + r) + M_2 \end{aligned}$$

   Notice that conceptually we can think of having money saved (or borrowed) from each of the first 2 periods. The money not spent in period 0 will earn interest for two periods, while the money not spend in period 1 will earn interest for just one period.

3. In the general t-period case, the same logic follows. It is easiest to write the equation using a summation operator, which after moving all consumption to the left hand side of the equation is:

$$\begin{aligned} c_0(1 + r)^t + c_1(1 + r)^{t-1} + ... + c_T &\leq M_0(1 + r)^T + M_1(1 + r)^{T-1} + ... + M_T \Leftrightarrow \\ c_0 + c_1(\frac{1}{1 + r}) + ... + c_T(\frac{1}{1 + r})^T &\leq M_0 + M_1(\frac{1}{1 + r}) + ... + M_T(\frac{1}{1 + r})^T \Leftrightarrow \\ \Sigma_{t=0}^T c_t(\frac{1}{1 + r})^t &\leq \Sigma_{t=0}^T M_t(\frac{1}{1 + r})^t \end{aligned}$$

Problem 3. (15 points)

Consider a version of the classical Condorcet paradox. Suppose we have three political candidates, A, B, and C, and that there are three voters with preferences as follows (candidates being listed in *decreasing* order of preference):

- Voter 1: A B C
- Voter 2: B C A
- Voter 3: C A B

Now define societal preferences over $\{A, B, C\}$ as follows: $x \succsim y$ if at least two voters prefer x to y. So, for example, $A \succsim B$ if at least two voters prefer A over B. Using this definition of the weak preference relation $\succsim$, prove (or show that it is false) that these preferences are: (i) complete; (ii) transitive. Be clear and complete in your claims. Can you provide a utility function that represents these preferences? (15 points)

Solution to Problem 3.

1. Is the societal preference relation ($\succsim$) complete? We say that a preference relation is complete if for any elements x,y in the set X we can say either $x \succsim y$ or $x \precsim y$ or both $x \succsim y$ and $x \precsim y$. So in this problem we have to determine whether we can write pairwise comparisons between A,B,C using our preference relation. From the definition of the particular societal preferences for this problem we have that: $A \succsim B$ and $B \succsim C$ and $C \succsim A$ (and $B \not\succsim A$ and $C \not\succsim B$ and $A \not\succsim C$). Since we can write the pairwise comparison between each element using the preference relation (as defined in this problem) then we conclude that the preference relation is complete.

2. Is the societal preference relation ($\succsim$) transitive? We say that a preference relation is transitive if for any elements x, y, z in the set X if $x \succsim y$ and $y \succsim z$ then $x \succsim z$. Importantly, a preference relation would not be transitive if for any elements x, y, z in the set X if $x \succsim y$ and $y \succsim z$ and $x \not\succsim z$. Note that since we allow for both $x \succsim z$ and $z \succsim x$ (over the same elements) then to show that a preference relation is not transitive we can't simply say that $z \succsim x$. In this problem we have that: $A \succsim B$ and $B \succsim C$ and $A \not\succsim C$. We can conclude that the preference relation is not transitive.

3. Can we write a utility function to represent the societal preferences? We can write a utility function if and only if the preference relation is both complete and transitive (remember we say that preferences are rational if the preference relation is complete and transitive). Since this preference relation is not transitive then there is no utility function that represents these preferences.