Econ 101A – Midterm 1
Th 28 February 2008.

You have approximately 1 hour and 20 minutes to answer the questions in the midterm. Vikram will collect the exams at 11.00 sharp. Show your work, and good luck!

Problem 1. Three-Good Cobb-Douglas. (50 points) Seung likes three goods: x_1, x_2, and x_3. He is aware that in Econ 101A we only use two goods, but he is too attached to all of them to let go of one. He maximizes the utility function

$$u(x_1, x_2, x_3) = x_1^{\alpha_1} x_2^{\alpha_2} x_3^{\alpha_3},$$

with $0 < \alpha_i < 1$ for $i = 1, 2, 3$. The consumption good x_i has price p_i (for $i = 1, 2, 3$) and the individual has total income M.

1. Compute the marginal utility of consumption with respect to good x_1, $\partial u(x_1, x_2, x_3)/\partial x_1$. (2 points)
2. What is the limit of the marginal utility for $x_1 \to 0$ and for $x_1 \to \infty$? Interpret the economic intuition behind this feature of this utility function. (5 points)
3. Write the budget constraint. (3 points)
4. Write the maximization problem of Seung. Seung wants to achieve the highest utility subject to the budget constraint. Write down the boundary constraints for x_1, x_2, x_3, and neglect them for now. (3 points)
5. Assuming that the budget constraint holds with equality, write down the Lagrangean and derive the first order conditions with respect to x_1, x_2, x_3, and λ. (5 points)
6. Solve for x_1^* as a function of the prices p_1, p_2, p_3, the income M, and the parameters α_1, α_2, and α_3. [Hint: combine the first and second first-order condition, then combine the first and third first-order condition, and finally plug in budget constraint] Similarly solve for x_2^* and x_3^*. (6 points)
7. Is this true or false? Show: "Cobb-Douglas preferences have the feature that the share of money spent on each good does not depend on the income, or on prices" (6 points)
8. Are the boundary conditions for x_1, x_2, and x_3 satisfied? (2 points)
9. Is good x_1 a normal good (for all values of M and prices p_i)? Compute and answer. (4 points)
10. Plot the implied demand function for x_1, that is plot x_1 as a function of p_1. (Put p_1 on the y axis and x_1 on the x axis) (4 points)
11. Is good x_1 a Giffen good? Why did you know this already from the answer to question 9? (5 points)
12. Are goods x_1 and x_2 gross complements, gross substitutes, or neither? Define and answer. (5 points)

Solution to Problem 1.

1. The marginal utility of consumption $\partial u(x_1, x_2, x_3)/\partial x_1$ is

$$\partial u(x_1, x_2, x_3)/\partial x_1 = \alpha_1 x_1^{\alpha_1 - 1} x_2^{\alpha_2} x_3^{\alpha_3}.$$

2. For $x_1 \to 0$ the marginal utility converges to $+\infty$ (keep in mind $\alpha_1 < 1$). This means that for very low consumption of x_1, the agent has un unlimited desire for some consumption of that good. For $x_1 \to \infty$ the marginal utility converges to 0. This means that for very high consumption of x_1, the additional unit of consumption has almost no added utility.
3. The budget constraint is

$$p_1 x_1 + p_2 x_2 + p_3 x_3 \leq M.$$

4. Seung maximizes

$$\begin{aligned}
\max_{x_1,x_2,x_3} u(x_1,x_2,x_3) &= x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3} \\
s.t. p_1x_1+p_2x_2+p_3x_3 &\le M \\
s.t.\ x_1 &\ge 0 \\
s.t.\ x_2 &\ge 0 \\
s.t.\ x_3 &\ge 0
\end{aligned}$$

5. The Lagrangean is

$$L(x_1,x_2,x_3,\lambda) = x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3} - \lambda(p_1x_1+p_2x_2+p_3x_3-M).$$

The first order conditions are

$$\begin{aligned}
f.o.c. \text{ with respect to } x_1 &: \quad \alpha_1 x_1^{\alpha_1-1}x_2^{\alpha_2}x_3^{\alpha_3} - \lambda p_1 = 0 \\
f.o.c. \text{ with respect to } x_1 &: \quad \alpha_2 x_1^{\alpha_1}x_2^{\alpha_2-1}x_3^{\alpha_3} - \lambda p_2 = 0 \\
f.o.c. \text{ with respect to } x_1 &: \quad \alpha_3 x_1^{\alpha_1}x_2^{\alpha_2}x_3^{\alpha_3-1} - \lambda p_3 = 0 \\
f.o.c. \text{ with respect to } \lambda &: \quad -(p_1x_1+p_2x_2+p_3x_3-M) = 0
\end{aligned}$$

6. From the first two f.o.c. we derive

$$\frac{\alpha_1}{\alpha_2}\frac{x_2}{x_1} = \frac{p_1}{p_2}.$$

which implies

$$x_2 = \frac{p_1}{p_2}\frac{\alpha_2}{\alpha_1}x_1.$$

From the first and third f.o.c. we derive

$$\frac{\alpha_1}{\alpha_3}\frac{x_3}{x_1} = \frac{p_1}{p_3}$$

which implies

$$x_3 = \frac{p_1}{p_3}\frac{\alpha_3}{\alpha_1}x_1.$$

Substituting the solutions for x_2 and x_3 in the budget constraint we obtain

$$p_1x_1 + p_2\left(\frac{p_1}{p_2}\frac{\alpha_2}{\alpha_1}x_1\right) + p_3\left(\frac{p_1}{p_3}\frac{\alpha_3}{\alpha_1}x_1\right) = M$$

which implies

$$x_1^* = \frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_3}\frac{M}{p_1}.$$

Using the expressions above for x_2 and x_3, we obtain

$$\begin{aligned}
x_2^* &= \frac{\alpha_2}{\alpha_1+\alpha_2+\alpha_3}\frac{M}{p_2} \text{ and} \\
x_3^* &= \frac{\alpha_3}{\alpha_1+\alpha_2+\alpha_3}\frac{M}{p_3}.
\end{aligned}$$

7. We can re-write the expressions above for x_1^*, x_2^* and x_3^* as follows:

$$\frac{x_i^* p_i}{M} = \frac{\alpha_1}{\alpha_1+\alpha_2+\alpha_3}$$

for $i = 1, 2$, and 3. The left-hand side is the share of money M spent on good i, and the right hand side of the equation shows that this is a constant, it does not depend on prices or income. Hence, the statement is true. This is a peculiar feature of Cobb-Douglas preferences.

8. In all of the expressions above for x_1^*, x_2^* and x_3^*, the conditions $x_i^* \geq 0$ are satisfied, and hence the boundary constraints are satisfied.

9. We can compute
$$\partial x_1^*/\partial M = \frac{\alpha_1}{\alpha_1 + \alpha_2 + \alpha_3} \frac{1}{p_1},$$
which is positive for all levels of the parameters, and hence the good is normal.

10. The demand function is a hyperbola, monotnically decreasing as we expect most demand functions to be.

11. Hence, the good in not a Giffen good. We knew this already from the fact that good x_1 is a normal good, that is, $\partial x_1^*/\partial M > 0$. Given the Slutzky equation, we know that a normal good can never be Giffen. Formally,
$$\frac{\partial x_1^*}{\partial p_1} = \frac{\partial h_1^*}{\partial p_1} - \frac{\partial x_1^*}{\partial M} x_i^*.$$
Since $\partial h_1^*/\partial p_1 < 0$ and $\partial x_1^*/\partial M > 0$ (normal good), we know that $\partial x_1^*/\partial p_1 < 0$.

12. Two goods are raw complements (substitutes) if $\partial x_1^*/\partial p_2 > 0$ (< 0). In this case, $\partial x_1^*/\partial p_2 = 0$ (and also $\partial x_2^*/\partial p_1 = 0$), hence the two goods are neither substitutes nor complements. This is a feature of Cobb-Douglas preferences.

Problem 2. (26 points)

1. Angela has utilty function $u(x_1, x_2) = 2x_1 + 2x_2$.

   (a) Plot the indifference curves of Angela. What kind of goods do they represent? (4 points)

   (b) Using the plot you did, find the utility-maximing solution x_1^*, x_2^* for prices $p_1 = 1$, $p_2 = 2$ and income M. Argue the steps you make. (8 points)

2. Kim has utility function $u(x_1, x_2) = \min(x_1, 2x_2)$

   (a) Plot the indifference curves of Kim. What kind of goods do they represent? (4 points)

   (b) Are the preferences represented by this utility function monotonic? Define. (4 points)

   (c) Are they strictly monotonic? Define. (6 points)

Solution to Problem 2.

1. Angela's preferences:

   (a) The indifference curves are straight lines with slope -1. The goods x_1 and x_2 are perfect substitutes, the individual only cares about the sum of the two goods. Do not be fooled by the 2 in front of the utility function, you can just divide the expression by 2 and get back the usual $u(x_1, x_2) = x_1 + x_2$.

   (b) Since good x_2 is more expensive, the individual will never purchase it, and will spend all the money on good x_1. Hence, the solutions are $x_1^* = M/p_1 = M$ and $x_2^* = 0$.

2. Kim's preferences:

   (a) The indifference curves are straight angles. The goods x_1 and x_2 are perfect complements, as for the case of left and right shoe, Kim only cares about x_1 if she also has enough of x_2.

   (b) The preferences are monotonic if $x_i \geq y_i$ for all i implies $x \succcurlyeq y$. These preferences are indeed monotonic. If $x_i \geq y_i$ for all i, then $\min(x_1, 2x_2) \geq \min(y_1, 2y_2)$.

   (c) The preferences are strongly monotonic if $x_i \geq y_i$ for all i and $x_j > y_j$ for some j implies $x \succ y$. Consider $x = (6, 2)$ and $y = (5, 2)$. Clearly, $x_i \geq y_i$ for all i and $x_j > y_j$ for some j, but $x \succ y$ does not hold, since $u(x) = 4 = u(y)$ and hence $x \sim y$.