

## Econ 101A – Problem Set 1

Due in class on Th February 2. No late Problem Sets accepted, sorry!

This Problem set tests the knowledge that you accumulated in the first 4 lectures. It is mostly on the mathematical techniques that we developed, but there is also some application to consumer decisions, as introduced in Lecture 4. General rules for problem sets: show your work, write down the steps that you use to get a solution (no credit for right solutions without explanation), write legibly. If you cannot solve a problem fully, write down a partial solution. We give partial credit for partial solutions that are correct. Do not forget to write your name on the problem set!

**Problem 1. Univariate unconstrained maximization.** (10 points) Consider the following maximization problem:

$$\max_x f(x; x_0) = \exp(-(x - x_0)^2)$$

- 1. Write down the first order conditions for this problem with respect to  $x$  (notice that  $x_0$  is a parameter, you should not maximize with respect to it). (1 point)
- 2. Solve explicitly for  $x^*$  that satisfies the first order conditions. (1 point)
- 3. Compute the second order conditions. Is the stationary point that you found in point 2 a maximum? Why (or why not)? (2 points)
- 4. As a comparative statics exercise, compute the change in  $x^*$  as  $x_0$  varies. In other words, compute  $dx^*/dx_0$ . (2 points)
- 5. We are interested in how the value function  $f(x^*(x_0); x_0)$  varies as  $x_0$  varies. We do it two ways. First, plug in  $x^*(x_0)$  from point 2 and then take the derivative with respect to  $x_0$ . Second, use the envelope theorem. You should get the same result! (2 points)
- 6. Is the function  $f$  concave in  $x$ ? (2 points)

**Problem 2. Multivariate unconstrained maximization.** (13 points) Consider the following maximization problem:

$$\max_{x,y} f(x, y; a, b) = ax^2 - x + by^2 - y$$

- 1. Write down the first order conditions for this problem with respect to  $x$  and  $y$  (notice that  $a$  and  $b$  are parameters, you do not need to maximize with respect to them). (1 point)
- 2. Solve explicitly for  $x^*$  and  $y^*$  that satisfy the first order conditions. (1 point)
- 3. Compute the second order conditions. Under what conditions for  $a$  and  $b$  is the stationary point that you found in point 2 a maximum? (2 points)
- 4. Assume that the conditions for  $a$  and  $b$  that you found in point 3 are met. As a comparative statics exercise, compute the change in  $y^*$  as  $a$  varies. In other words, compute  $dy^*/da$ . Compute it both directly using the solution that you obtained in point 2 and using the general method presented in class that makes use of the implicit function theorem. The two results should coincide! (3 points)
- 5. We are interested in how the value function  $f(x^*(a, b); y^*(a, b))$  varies as  $a$  varies. We do it two ways. First, plug in  $x^*(a, b)$  and  $y^*(a, b)$  from point 2 into  $f$  and then take the derivative of  $f(x^*(a, b); y^*(a, b))$  with respect to  $a$ . Second, use the envelope theorem. You should get the same result! Which method is faster? (3 points)
- 6. Under what conditions on  $a$  and  $b$  is the function  $f$  concave in  $x$  and  $y$ ? When is it convex in  $x$  and  $y$ ? (3 points)

**Problem 3. Multivariate constrained maximization.** (19 points) Consider the following maximization problem:

$$\begin{aligned} \max_{x,y} u(x,y) &= x^\alpha y^\beta \\ \text{s.t. } p_x x + p_y y &= M, \end{aligned}$$

with  $0 < \alpha < 1$ ,  $0 < \beta < 1$ . The problem above is a classical maximization of utility subject to a budget constraint. The utility function  $x^\alpha y^\beta$  is also called a Cobb-Douglas utility function. You can interpret  $p_x$  as the price of good  $x$  and  $p_y$  as the price of good  $y$ . Finally,  $M$  is the total income. I provide these details to motivate this problem. In order to solve the problem, you only need to apply the theory of constrained maximization that we covered in class. But beware, you are going to see a lot more Cobb-Douglas functions in the next few months!

- 1. Write down the Lagrangean function (1 point)
- 2. Write down the first order conditions for this problem with respect to  $x$ ,  $y$ , and  $\lambda$ . (1 point)
- 3. Solve explicitly for  $x^*$  and  $y^*$  as a function of  $p_x, p_y, M, \alpha$ , and  $\beta$ . (3 points)
- 4. Notice that the utility function  $x^\alpha y^\beta$  is defined only for  $x > 0, y > 0$ . Does your solution for  $x^*$  and  $y^*$  satisfies these constraints? What assumptions you need to make about  $p_x, p_y$  and  $M$  so that  $x^* > 0$  and  $y^* > 0$ ? (1 point)
- 5. Write down the bordered Hessian. Compute the determinant of this 3x3 matrix and check that it is positive (this is the condition that you need to check for a constrained maximum) (3 points)
- 6. As a comparative statics exercise, compute the change in  $x^*$  as  $p_x$  varies. In order to do so, use directly the expressions that you obtained in point 3, and differentiate  $x^*$  with respect to  $p_x$ . Does your result make sense? That is, what happens to the quantity of good  $x^*$  consumed as the price of good  $x$  increases? (2 points)
- 7. Similarly, compute the change in  $x^*$  as  $p_y$  varies. Does this result make sense? What happens to the quantity of good  $x^*$  consumed as the price of good  $y$  increases? (2 points)
- 8. Finally, compute the change in  $x^*$  as  $M$  varies. Does this result make sense? What happens to the quantity of good  $x^*$  consumed as the total income  $M$  increases? (2 points)
- 9. We have so far looked at the effect of changes in  $p_x, p_y$ , and  $M$  on the quantities of goods consumed. We now want to look at the effects on the utility of the consumer at the optimum. Use the envelope theorem to calculate  $du(x^*(p_x, p_y, M), y^*(p_x, p_y, M))/dp_x$ . What happens to utility at the optimum as the price of good  $x$  increases? Is this result surprising? (2 points)
- 10. Use the envelope theorem to calculate  $\partial u(x^*(p_x, p_y, M), y^*(p_x, p_y, M))/\partial M$ . What happens to utility at the optimum as total income  $M$  increases? Is this result surprising? (2 points)

**Problem 4. Rationality of preferences** (5 points) Prove the following statements:

- if  $\succeq$  is rational, then  $\sim$  is transitive, that is,  $x \sim y$  and  $y \sim z$  implies  $x \sim z$  (3 points)
- if  $\succeq$  is rational, then  $\succeq$  has the reflexive property, that is,  $x \succeq x$  for all  $x$ . (2 points)