Outline

1. Implicit Function Theorem

2. Envelope Theorem

3. Convexity and concavity

4. Constrained Maximization
1 Implicit function theorem

- Implicit function: Ch. 2, pp. 31-32

- Consider function $x_2 = g(x_1, p)$

- Can rewrite as $x_2 - g(x_1, p) = 0$

- **Implicit function** has form: $h(x_2, x_1, p) = 0$

- Often we need to go from implicit to explicit function

- Example 3: $1 - x_1 * x_2 - e^{x_2} = 0$.

- Write $x_1$ as function of $x_2$:

- Write $x_2$ as function of $x_1$: 
• **Univariate implicit function theorem (Dini):** Consider an equation \( f(p, x) = 0 \), and a point \((p_0, x_0)\) solution of the equation. Assume:

1. \( f \) continuously differentiable in a neighbourhood of \((p_0, x_0)\);
2. \( f_x(p_0, x_0) \neq 0 \).

• Then:

1. There is one and only function \( x = g(p) \) defined in a neighbourhood of \( p_0 \) that satisfies \( f(p, g(p)) = 0 \) and \( g(p_0) = x_0 \);
2. The derivative of \( g(p) \) is

\[
g'(p) = -\frac{f_p(p, g(p))}{f_x(p, g(p))}
\]
• Example 3 (continued): $1 - x_1 \cdot x_2 - e^{x_2} = 0$

• Find derivative of $x_2 = g(x_1)$ implicitly defined for $(x_1, x_2) = (1, 0)$

• Assumptions:
  1. Satisfied?
  2. Satisfied?

• Compute derivative
Multivariate implicit function theorem (Dini):
Consider a set of equations \( f_1(p_1, \ldots, p_n; x_1, \ldots, x_s) = 0; \ldots; f_s(p_1, \ldots, p_n; x_1, \ldots, x_s) = 0 \), and a point \((p_0, x_0)\) solution of the equation. Assume:

1. \( f_1, \ldots, f_s \) continuously differentiable in a neighbourhood of \((p_0, x_0)\);

2. The following Jacobian matrix \( \frac{\partial f}{\partial x} \) evaluated at \((p_0, x_0)\) has determinant different from 0:

\[
\frac{\partial f}{\partial x} = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_s} \\
... & ... \\
\frac{\partial f_s}{\partial x_1} & \frac{\partial f_s}{\partial x_s}
\end{pmatrix}
\]
Then:

1. There is one and only set of functions $x \leftarrow g(p)$ defined in a neighbourhood of $p_0$ that satisfy $f(p, g(p)) = 0$ and $g(p_0) = x_0$;

2. The partial derivative of $x_i$ with respect to $p_k$ is

$$\frac{\partial g_i}{\partial p_k} = -\frac{\det \left( \frac{\partial (f_1, \ldots, f_s)}{\partial (x_1, \ldots x_{i-1}, p_k, x_{i+1}, \ldots, x_s)} \right)}{\det \left( \frac{\partial f}{\partial x} \right)}$$
• Example 2 (continued): Max $h(x_1, x_2) = p_1 x_1^2 + p_2 x_2^2 - 2x_1 - 5x_2$

• f.o.c. $x_1 : 2p_1 x_1 - 2 = 0 = f_1(p,x)$

• f.o.c. $x_2 : 2p_2 x_2 - 5 = 0 = f_2(p,x)$

• Comparative statics of $x_1^*$ with respect to $p_1$?

• First compute $\det \left( \frac{\partial f}{\partial x} \right)$

$$
\begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{pmatrix}
= 
\begin{pmatrix}
\phantom{0} \\
\phantom{0}
\end{pmatrix}
$$
• Then compute \( \det \left( \frac{\partial (f_1, \ldots, f_s)}{\partial (x_1, \ldots, x_{i-1}, x_i, p_k, x_{i+1}, \ldots, x_s)} \right) \)

\[
\begin{pmatrix}
\frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial p_1} & \frac{\partial f_2}{\partial x_2}
\end{pmatrix} = \begin{pmatrix} \text{(fill in)} \end{pmatrix}
\]

• Finally, \( \frac{\partial x_1}{\partial p_1} = \)

• Why did you compute \( \det \left( \frac{\partial f}{\partial x} \right) \) already?
2 Envelope Theorem

- Ch. 2, pp. 35-39

- You now know how $x_1^*$ varies if $p_1$ varies.

- How does $h(x^*(p))$ vary as $p_1$ varies?

- Differentiate $h(x_1^*(p_1, p_2), x_2^*(p_1, p_2), p_1, p_2)$ with respect to $p_1$:

  $\frac{dh(x_1^*(p_1, p_2), x_2^*(p_1, p_2), p_1, p_2)}{dp_1} = \left( \frac{\partial h(x^*, p)}{\partial x_1} \times \frac{dx_1^*(p)}{dp_1} \right) + \left( \frac{\partial h(x^*, p)}{\partial x_2} \times \frac{dx_2^*(p)}{dp_1} \right) + \frac{\partial h(x^*, p)}{dp_1}$

- The first two terms are zero.
• **Envelope Theorem** for unconstrained maximization. Assume that you maximize function $f(x; p)$ with respect to $x$. Consider then the function $f$ at the optimum, that is, $f(x^*(p), p)$. The total differential of this function with respect to $p_i$ equals the partial derivative with respect to $p_i$:

$$
\frac{df(x^*(p), p)}{dp_i} = \frac{\partial f(x^*(p), p)}{\partial p_i}.
$$

• You can disregard the indirect effects. Graphical intuition.
3 Convexity and concavity

• Function $f$ from $C \subset \mathbb{R}^n$ to $\mathbb{R}$ is concave if

$$f(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y)$$

for all $x, y \in C$ and for all $t \in [0, 1]$

• Notice: $C$ must be convex set, i.e., if $x \in C$ and $y \in C$, then $tx + (1 - t)y \in C$, for $t \in [0, 1]$

• Function $f$ from $C \subset \mathbb{R}^n$ to $\mathbb{R}$ is strictly concave if

$$f(tx + (1 - t)y) > tf(x) + (1 - t)f(y)$$

for all $x, y \in C$ and for all $t \in (0, 1)$

• Function $f$ from $\mathbb{R}^n$ to $\mathbb{R}$ is convex if $-f$ is concave.
• Alternative characterization of convexity.

• A function $f$, twice differentiable, is concave if and only if for all $x$ the subdeterminants $|H_i|$ of the Hessian matrix have the property $|H_1| \leq 0, |H_2| \geq 0, |H_3| \leq 0$, and so on.

• For the univariate case, this reduces to $f'' \leq 0$ for all $x$

• For the bivariate case, this reduces to $f_{x,x}'' \leq 0$ and $f_{x,x}'' \cdot f_{y,y}'' - (f_{x,y}'')^2 \geq 0$

• A twice-differentiable function is strictly concave if the same property holds with strict inequalities.
• Examples.

1. For which values of $a$, $b$, and $c$ is $f(x) = ax^3 + bx^2 + cx + d$ is the function concave over $R$? Strictly concave? Convex?

2. Is $f(x, y) = -x^2 - y^2$ concave?

• For Example 2, compute the Hessian matrix

$$
H = \begin{pmatrix}
    f''_{x,x} &=& f''_{x,y} \\
    f''_{y,x} &=& f''_{y,y}
\end{pmatrix}
$$

• Compute $|H_1| = f''_{x,x}$ and $|H_2| = f''_{x,x} \cdot f''_{y,y} - (f''_{x,y})^2$
Why are convexity and concavity important?

**Theorem.** Consider a twice-differentiable concave (convex) function over \( C \subset R^n \). If the point \( x_0 \) satisfies the first order conditions, it is a global maximum (minimum).

For the proof, we need to check that the second-order conditions are satisfied.

These conditions are satisfied by definition of concavity!

(We have only proved that it is a local maximum)
4 Constrained maximization

• Ch. 2, pp. 39-45

• So far unconstrained maximization on \( R \) (or open subsets)

• What if there are constraints to be satisfied?

• Example 1: \( \max_{x,y} x \cdot y \) subject to \( 3x + y = 5 \)

• Substitute it in: \( \max_{x,y} x \cdot (5 - 3x) \)

• Solution: \( x^* = \)

• Example 2: \( \max_{x,y} xy \) subject to \( x \exp(y) + y \exp(x) = 5 \)

• Solution: ?
• Graphical intuition on general solution.

• Example 3: \( \max_{x,y} f(x, y) = x \ast y \) s.t. \( h(x, y) = x^2 + y^2 - 1 = 0 \)

• Draw 0 = \( h(x, y) = x^2 + y^2 - 1 \).

• Draw \( x \ast y = K \) with \( K > 0 \). Vary \( K \)

• Where is optimum?

• Where \( dy/dx \) along curve \( xy = K \) equals \( dy/dx \) along curve \( x^2 + y^2 - 1 = 0 \)

• Write down these slopes.
• Idea: Use implicit function theorem.

• Heuristic solution of system

\[
\max_{x,y} f(x, y) \\
\text{s.t. } h(x, y) = 0
\]

• Assume:

  – continuity and differentiability of \( h \)

  – \( h'_y \neq 0 \) (or \( h'_x \neq 0 \))

• Implicit function Theorem: Express \( y \) as a function of \( x \) (or \( x \) as function of \( y \))!
• Write system as $\max_x f(x, g(x))$

• f.o.c.: $f'_x(x, g(x)) + f'_y(x, g(x)) \cdot \frac{\partial g(x)}{\partial x} = 0$

• What is $\frac{\partial g(x)}{\partial x}$?

• Substitute in and get: $f'_x(x, g(x)) + f'_y(x, g(x)) \cdot (-h'_{x}/h'_{y}) = 0$ or

\[
\frac{f'_x(x, g(x))}{f'_y(x, g(x))} = \frac{h'_x(x, g(x))}{h'_y(x, g(x))}
\]
• **Lagrange Multiplier Theorem, necessary condition.** Consider a problem of the type

\[
\max_{x_1,\ldots,x_n} f(x_1, x_2, \ldots, x_n; p)
\]

subject to

\[
\begin{align*}
  h_1(x_1, x_2, \ldots, x_n; p) &= 0 \\
  h_2(x_1, x_2, \ldots, x_n; p) &= 0 \\
  \quad &\vdots \\
  h_m(x_1, x_2, \ldots, x_n; p) &= 0
\end{align*}
\]

with \( n > m \). Let \( x^* = x^*(p) \) be a local solution to this problem.

• Assume:

  - \( f \) and \( h \) differentiable at \( x^* \)
  
  - the following Jacobian matrix at \( x^* \) has maximal rank

\[
J = \begin{pmatrix}
\frac{\partial h_1}{\partial x_1}(x^*) & \cdots & \frac{\partial h_1}{\partial x_n}(x^*) \\
\vdots & \ddots & \vdots \\
\frac{\partial h_m}{\partial x_1}(x^*) & \cdots & \frac{\partial h_m}{\partial x_n}(x^*)
\end{pmatrix}
\]
• Then, there exists a vector $\lambda = (\lambda_1, \ldots, \lambda_m)$ such that $(x^*, \lambda)$ maximize the Lagrangean function

$$L(x, \lambda) = f(x; p) - \sum_{j=0}^{m} \lambda_j h_j(x; p)$$

• Case $n = 2, m = 1$.

• First order conditions are

$$\frac{\partial f(x; p)}{\partial x_i} - \lambda \frac{\partial h(x; p)}{\partial x_i} = 0$$

for $i = 1, 2$

• Rewrite as

$$\frac{f'_{x_1}}{f'_{x_2}} = \frac{h'_{x_1}}{h'_{x_2}}$$
Constrained Maximization, Sufficient condition for the case \( n = 2, m = 1 \).

- If \( \mathbf{x}^* \) satisfies the Lagrangean condition, and the determinant of the bordered Hessian

\[
H = \begin{pmatrix}
0 & -\frac{\partial h}{\partial x_1}(\mathbf{x}^*) & -\frac{\partial h}{\partial x_2}(\mathbf{x}^*) \\
-\frac{\partial h}{\partial x_1}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_1^2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_2 \partial x_1}(\mathbf{x}^*) \\
-\frac{\partial h}{\partial x_2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_1 \partial x_2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_2^2}(\mathbf{x}^*)
\end{pmatrix}
\]

is positive, then \( \mathbf{x}^* \) is a constrained maximum.

- If it is negative, then \( \mathbf{x}^* \) is a constrained minimum.

- Why? This is just the Hessian of the Lagrangean \( L \) with respect to \( \lambda, x_1, \) and \( x_2 \).
Example 4: $\max_{x,y} x^2 - xy + y^2$ s.t. $x^2 + y^2 - p = 0$

$\max_{x,y,\lambda} x^2 - xy + y^2 - \lambda(x^2 + y^2 - p)$

F.o.c. with respect to $x$:

F.o.c. with respect to $y$:

F.o.c. with respect to $\lambda$:

Candidates to solution?

Maxima and minima?
Next Class

Next class:

- More on Constrained Maximization
- Preferences