Economics 101A
(Lecture 2)

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Outline

1. Optimization with 1 variable

2. Multivariate optimization

3. Comparative Statics

4. Implicit function theorem
1 Optimization with 1 variable

- Nicholson, Ch.2, pp. 20-23

- Example. Function $y = -x^2$ – Graph it

- What is the maximum?

- Maximum is at 0

- General method?
• Sure! Use derivatives

• Derivative is slope of the function at a point:

\[ \frac{\partial f(x)}{\partial x} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} \]

• Necessary condition for maximum \( x^* \) is

\[ \frac{\partial f(x^*)}{\partial x} = 0 \quad (1) \]

• Try with \( y = -x^2 \).

\[ \frac{\partial f(x)}{\partial x} = \quad = 0 \quad \Rightarrow \quad x^* = \]
• Does this guarantee a maximum? No!

• Consider the function \( y = x^3 \)

\[
\frac{\partial f(x)}{\partial x} = \quad = 0 \quad \implies \quad x^* =
\]

• Plot \( y = x^3 \).
• **Sufficient condition for a (local) maximum:**

\[
\frac{\partial f(x^*)}{\partial x} = 0 \quad \text{and} \quad \frac{\partial^2 f(x)}{\partial^2 x} \bigg|_{x^*} < 0 \quad (2)
\]

• **Proof:** At a maximum, \( f(x^* + h) - f(x^*) < 0 \) for all \( h \).

• **Taylor Rule:** \[ f(x^* + h) - f(x^*) = \frac{\partial f(x^*)}{\partial x} h + \frac{1}{2} \frac{\partial^2 f(x^*)}{\partial^2 x} h^2 + \text{higher order terms.} \]

• **Notice:** \( \frac{\partial f(x^*)}{\partial x} = 0 \).

• \( f(x^* + h) - f(x^*) < 0 \) for all \( h \) \( \implies \) \( \frac{\partial^2 f(x^*)}{\partial^2 x} h^2 < 0 \)

\[ 0 \implies \frac{\partial^2 f(x^*)}{\partial^2 x} < 0 \]

• **Careful:** Maximum may not exist: \( y = \exp(x) \)
• Tricky examples:

  - *Minimum.* $y = x^2$

  - *No maximum.* $y = \exp(x)$ for $x \in (-\infty, +\infty)$

  - *Corner solution.* $y = x$ for $x \in [0, 1]$
2 Multivariate optimization

- Nicholson, Ch.2, pp. 26-31 and 33-35

- Function from $R^n$ to $R$: $y = f(x_1, x_2, \ldots, x_n)$

- Partial derivative with respect to $x_i$:

$$
\frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots x_n) - f(x_1, \ldots, x_i, \ldots x_n)}{h}
$$

- Slope along dimension $i$

- Total differential:

$$
df = \frac{\partial f(x)}{\partial x_1} dx_1 + \frac{\partial f(x)}{\partial x_2} dx_2 + \ldots + \frac{\partial f(x)}{\partial x_n} dx_n
$$
• One important economic example

• Example 1: Partial derivatives of \( y = f(L, K) = L^{0.5}K^{0.5} \)

  • \( f'_L = \)  
    (marginal productivity of labor)

  • \( f'_K = \)  
    (marginal productivity of capital)

  • \( f''_{L,K} = \)
Maximization over an open set (like $\mathbb{R}$)

- **Necessary condition for maximum** $x^*$ is

$$\frac{\partial f(x^*)}{\partial x_i} = 0 \quad \forall i$$  \hspace{1cm} (3)

or in vectorial form

$$\nabla f(x) = 0$$

- These are commonly referred to as first order conditions (f.o.c.)

- Sufficient conditions? Define Hessian matrix $H$:

$$H = \begin{pmatrix} f''_{x_1,x_1} & f''_{x_1,x_2} & \cdots & f''_{x_1,x_n} \\ \vdots & \ddots & \vdots & \vdots \\ f''_{x_n,x_1} & f''_{x_n,x_2} & \cdots & f''_{x_n,x_n} \end{pmatrix}$$
• Subdeterminant $|H|_i$ of Matrix $H$ is defined as the determinant of submatrix formed by first $i$ rows and first $i$ columns of matrix $H$.

• Examples.

  - $|H|_1$ is determinant of $f''_{x_1,x_1}$, that is, $f''_{x_1,x_1}$

  - $|H|_2$ is determinant of

    $$H = \begin{pmatrix} f''_{x_1,x_1} & f''_{x_1,x_2} \\ f''_{x_2,x_1} & f''_{x_2,x_2} \end{pmatrix}$$

• **Sufficient condition for maximum $x^*$**.

  1. $x^*$ must satisfy first order conditions;

  2. Subdeterminants of matrix $H$ must have alternating signs, with subdeterminant of $H_1$ negative.
• Case with \( n = 2 \)

• Condition 2 reduces to \( f''_{x_1,x_1} < 0 \) and \( f''_{x_1,x_1} f''_{x_2,x_2} - (f''_{x_1,x_2})^2 > 0 \).

• Example 2: \( h(x_1, x_2) = p_1 x_1^2 + p_2 x_2^2 - 2x_1 - 5x_2 \)

  • First order condition w/ respect to \( x_1 \)?

  • First order condition w/ respect to \( x_2 \)?

  • \( x_1^*, x_2^* = \)

  • For which \( p_1, p_2 \) is it a maximum?

  • For which \( p_1, p_2 \) is it a minimum?
3 Comparative statics

- Economics is all about ‘comparative statics’

- What happens to optimal economic choices if we change one parameter?

- Example: Car production. Consumer:
  1. Car purchase and increase in oil price
  2. Car purchase and increase in income

- Producer:
  1. Car production and minimum wage increase
  2. Car production and decrease in tariff on Japanese cars

- Next two sections
4 Implicit function theorem

- Implicit function: Ch. 2, pp. 31-32

- Consider function $x_2 = g(x_1, p)$

- Can rewrite as $x_2 - g(x_1, p) = 0$

- **Implicit function** has form: $h(x_2, x_1, p) = 0$

- Often we need to go from implicit to explicit function

- Example 3: $1 - x_1 \times x_2 - e^{x_2} = 0$.

- Write $x_1$ as function of $x_2$:

- Write $x_2$ as function of $x_1$: 
Univariate implicit function theorem (Dini): Consider an equation \( f(p, x) = 0 \), and a point \((p_0, x_0)\) solution of the equation. Assume:

1. \( f \) continuously differentiable in a neighbourhood of \((p_0, x_0)\); 
2. \( f_x(p_0, x_0) \neq 0 \).

Then:

1. There is one and only function \( x = g(p) \) defined in a neighbourhood of \( p_0 \) that satisfies \( f(p, g(p)) = 0 \) and \( g(p_0) = x_0 \);
2. The derivative of \( g(p) \) is 
   \[
   g'(p) = -\frac{f'_p(p, g(p))}{f'_x(p, g(p))}
   \]
• Example 3 (continued): \( 1 - x_1 \cdot x_2 - e^{x_2} = 0 \)

• Find derivative of \( x_2 = g(x_1) \) implicitly defined for \( (x_1, x_2) = (1, 0) \)

• Assumptions:
  1. Satisfied?
  2. Satisfied?

• Compute derivative
5 Next Class

• Next class:
  – Implicit Function Theorem II
  – Envelope Theorem
  – Convexity and Concavity
  – Constrained Maximization
  – Envelope Theorem II

• Going toward:
  – Preferences
  – Utility Maximization (where we get to apply maximization techniques the first time)