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Economics 202A
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OPTIMAL PORTFOLIOS AND THE MUTUAL FUND SEPARATION THEOREM

I. Set-up

There are two periods. In the first, an agent has wealth W that he or she allocates among assets. In the second, the assets' returns are realized and the agent consumes the proceeds.

There is a riskless asset, which pays a return of zero for sure, and N risky assets. Letting B denote the agent's purchases of the riskless asset and x_i his or her purchases of risky asset i, the budget constraint is $B + x_1 + x_2 + \ldots + x_N = W$. B and the x_i can be negative; that is, the agent can "sell short" assets.

Asset i pays return r_i. The r's have means $\mu_1, \mu_2, \ldots, \mu_N$ and variance-covariance matrix $\sum$. The matrix $\sum$ is N x N, and we denote its ij element by σ_{ij}. Thus, the agent's consumption is $C = B + x_1(1 + r_1) + x_2(1 + r_2) + \ldots + x_N(1 + r_N)$. We can use the budget constraint to rewrite this as

$$C = W + x_1r_1 + x_2r_2 + \ldots + x_Nr_N.$$

The agent's objective function is assumed to depend only on the mean and variance of consumption (and to be increasing in the mean and decreasing in the variance). One case where this arises is quadratic utility, as in class. Another case is constant absolute risk aversion utility and normally distributed returns, along the lines of Problem 8.14 in the book.

Since the individual cares only about the mean and variance of consumption, then the optimal allocation of the individual's wealth will have the lowest possible variance of consumption given its mean. (If not, it would be possible to lower the variance without changing the mean, which would make the agent better off.) Thus, rather the solving the full optimization problem, we will focus on minimizing variance for a given mean.

II. Case 1: Two risky assets, solved with algebra

With two risky assets, $C = W + x_1r_1 + x_2r_2$. Thus the mean of C is $C = W + x_1\mu_1 + x_2\mu_2$. To find the variance of C, note that $C – E[C] = (r_1 – \mu_1)x_1 + (r_2 – \mu_2)x_2$. Thus,

$$E[(C - E[C])^2] = E[((r_1 - \mu_1)x_1 + (r_2 - \mu_2)x_2)^2]$$

$$= x_1^2E[(r_1 - \mu_1)^2] + x_2^2E[(r_2 - \mu_1)^2] + 2x_1x_2E[(r_1 - \mu_1)(r_2 - \mu_2)]$$

$$= x_1^2\sigma_{11} + x_2^2\sigma_{22} + 2x_1x_2\sigma_{12}.$$

So the Lagrangian for the problem of minimizing variance subject to achieving some target level of expected consumption, Z, is

$$L = x_1^2\sigma_{11} + x_2^2\sigma_{22} + 2x_1x_2\sigma_{12} + \lambda[Z - (W + x_1\mu_1 + x_2\mu_2)].$$

The first-order conditions for x_1 and x_2 are

$$2x_1^*\sigma_{11} + 2x_2^*\sigma_{12} = \lambda\mu_1,$$

$$2x_2^*\sigma_{22} + 2x_1^*\sigma_{12} = \lambda\mu_2.$$

Solving these two linear equations for x_1^* and x_2^* gives us

$$(1) \quad x_1^* = \frac{\sigma_{22}\mu_1 - \sigma_{12}\mu_2}{\sigma_{11}\sigma_{22} - \sigma_{12}^2}\frac{\lambda}{2},$$

$$(2) \quad x_2^* = \frac{\sigma_{11}\mu_2 - \sigma_{12}\mu_1}{\sigma_{11}\sigma_{22} - \sigma_{12}^2}\frac{\lambda}{2}.$$

III. <u>Implications</u>

A. <u>The Determinants of Purchases the Two Risky Assets</u>

Equations (1) and (2) show what determines how much of the two assets the agent buys.

For example, suppose $\sigma_{12} = 0$. Then $x_1 = \lambda\mu_1/2\sigma_{11}$, $x_2 = \lambda\mu_2/2\sigma_{22}$. Thus whether the agent buys a positive, negative, or zero amount of the asset is determined by whether the asset's expected excess return is positive, negative, or zero. Purchases of an asset are proportional to its expected excess return, inversely proportional to its variance, and increasing in the importance the agent attaches to the mean of his or her consumption relative to its variance.

Note that from the point of view of the individual, whether the risk associated with an asset is idiosyncratic or systematic is endogenous. Continue to consider the case $\sigma_{12} = 0$. Then the covariance of the return on asset 1 with the agent's consumption is $x_1\sigma_{11}$; thus it depends on how much of the asset the agent buys. If we think of the agent as initially buying none of the asset and then gradually increasing his or her purchases, initially the covariance of the asset's return with the agent's consumption is zero, and so the asset's risk is idiosyncratic. But as the agent's purchases increase, the covariance of the return with the agent's consumption rises, and so the risk becomes increasingly systematic. If the expected return on the asset is strictly positive ($\mu_1 > 0$, and recall that we are considering the case of $\sigma_{12} = 0$), the agent will definitely buy a strictly positive amount of the asset (because initially there is only idiosyncratic risk). But the fact that the systematic risk associated with the asset increases as the agent's purchases increase means that he or she will not buy an arbitrarily large amount.

To see how the covariance between the returns on the two assets affects the agent's asset purchases, consider a small increase in σ_{12} starting from $\sigma_{12} = 0$. The marginal effect is to reduce the agent's purchases of asset 1 if his or her holdings of asset 2 are positive, and to increase his or her purchases of asset 1 if his or her holdings of asset 2 are negative.

B. The Mutual Fund Separation Theorem

Notice what happens as λ changes – that is, as the agent puts more or less weight on the mean relative to the variance. x_1 and x_2 change in the same proportion. Thus, agents who differ in their attitudes toward risk will buy different amounts of the riskless asset, but their mix of the risky assets (that is, their ratio of x_1 to x_2) will be the same.

This is the Mutual Fund Separation Theorem. We can construct an optimal mix (that is, an optimal mutual fund) of risky assets. Depending on their risk preferences, agents will choose different combinations of the safe asset and the mutual fund; but they will not choose different mixes of risky assets.

C. An Attempt at Intuition for the Mutual Fund Separation Theorem

Consider a portfolio of risky assets. If the agent buys none of the portfolio and puts all his or her wealth into the safe asset, his or her mean consumption is W, and its standard deviation is zero. As the agent moves out of the riskless asset into the portfolio, both the mean and standard deviation of consumption change linearly with the amount invested in the portfolio. Thus, as the agent shifts out of the riskless asset into the portfolio, the mean and standard deviation of his or her consumption move along a ray (in standard deviation of consumption–mean consumption space) from the point (0,W). Thus, each portfolio gives the agent access to a ray of points out of (0,W) in standard deviation–mean space. Every agent prefers to be on a higher ray than to be on a lower one. So every agent chooses the portfolio with the highest slope in this space – that is, the portfolio with the highest ratio of expected excess return (that is, the expected return on the portfolio minus that on the safe asset) to standard deviation. Agents' risk attitudes then determine where on the line they choose to be – that is, how much of the portfolio they buy.

IV. Case 2: N risky assets, solved with linear algebra

Here $C = W + X'r$, where $X = [x_1\ x_2\ \dots\ x_N]'$ and $r = [r_1\ r_2\ \dots\ r_N]'$. Thus, $E[C] = W + X'\mu$ (where $\mu = [\mu_1\ \mu_2\ \dots\ \mu_N]'$), and $Var(C) = X'\sum X$.

The Lagrangian is $L = X'\sum X + \lambda[Z - (W + X'\mu)]$. The derivative of $X'\sum X$ with respect to X is $(X'\sum)' + \sum X$, which equals $\sum' X + \sum X$, or $2\sum X$. (To see this, consider the derivative with respect to x_1. x1 appears twice in $X'\sum X$, and so there are two terms. The first term is $X'\sum$ times the vector $[1\ 0\ 0\ \dots\ 0]$, which is X' times $[\sigma_{11}\ \sigma_{21}\ \dots\ \sigma_{N1}]$, which is the first element of $\sum' X$ (and hence the first element of $\sum X$, since $\sum$ is symmetric). The second term is the vector $[1\ 0\ 0\ \dots\ 0]'$ times $\sum X$, which is $[\sigma_{11}\ \sigma_{12}\ \dots\ \sigma_{1N}]'$ times X, which is also the first element of $\sum X$. Thus the derivative of $X'\sum X$ with respect to x_1 is 2 times the first element of $\sum X$.)

Thus, the Nx1 vector of first-order conditions is $2\sum X^* = \lambda\mu$, which implies

$$X^* = \frac{\lambda}{2}\Sigma^{-1}\mu.$$

The Mutual Fund Separation Theorem holds here: when λ changes, all the elements of X^* change by the same proportion. The intuition (such as it is) is the same as for the case of two assets.