University of California
Department of Economics

Economics 202A
Fall 2015
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Problem Set 11
Due in Lecture Tuesday Nov. 24

1 Romer 9.13

2 Non Convex Adjustment Costs and Investment

Consider the model of investment with non-convex adjustment costs we analyzed in class. A firm's profit function take the form $\Pi(K,\theta) = K^\alpha \theta$ where θ captures all the shocks to the firm's profits and α is related to the market power of the firm. Denote r the risk free rate, assume there is no depreciation and that price of capital is $p_K = 1$, so r is also the user cost.

1. Suppose there are no adjustment costs to investment. Characterize the desired stock of capital K^* as a function of r, θ and α.

2. Define $Z = K/K^*$ as the *capital gap*. Show that we can write the firm's profits as:

$$\Pi(Z, K^*) = \frac{r}{\alpha} Z^\alpha K^*$$

3. Assume that θ follows a *Geometric* Brownian Motion of the form:

$$d\theta_t = \tilde{\mu}\theta_t dt + \sigma\theta_t d\omega_t$$

   where ω_t is a Brownian Motion. Using Itô's lemma, show that when the capital stock K is constant $z = \ln Z$ follows the following stochastic process:

$$dz = -\mu/(1-\alpha)dt - \sigma/(1-\alpha)d\omega_t$$

   where $\mu = \tilde{\mu} - \sigma^2/2$. [Hint: you can check, using Itô's lemma, that if θ follows a Geometric Brownian Motion with drift $\tilde{\mu}\theta$ and volatility $\sigma\theta$, then $\ln\theta$ follows an arithmetic brownian motion with drift $\tilde{\mu} - \sigma^2/2$ and volatility σ.]

4. Now suppose that the firm faces *non-convex adjustment costs*. We know that this will create a *range of inaction* $L \leq z \leq U$ where the firm will not adjust its capital stock. Define $v(z_t)$ the value function of the firm's problem per unit of desired capital in that inaction range. That is, $v(z)$ is defined as:

$$v(z_t) = E_t\left[\int_t^\infty e^{-r(s-t)}\frac{r}{\alpha}e^{\alpha z_s} ds\right]$$

where z_s follows the stochastic process derived above. Write the Hamiton-Jacobi-Bellman equation that $v(z)$ must satisfy and show that it is of the form:

$$rv(z) = \frac{r}{\alpha}e^{\alpha z} + Cv'(z) + Dv''(z)$$

for coefficients C and D that you will characterize.

5. We are going to solve this second order differential equation. The solution method consists in first finding the solutions of the *homogenous* equation, i.e. solutions to the equation

$$rv(z) = Cv'(z) + Dv''(z)$$

that ignores the 'forcing' term $(r/\alpha)e^{\alpha z}$. Given that the coefficients of this equation are constant, we are going to look for solutions of the form $v(z) = e^{\lambda z}$. Show that λ must satisfy the following characteristic equation:

$$\Phi(\lambda) = \lambda^2/2(\sigma/(1-\alpha))^2 - \mu/(1-\alpha)\lambda - r = 0$$

The homogenous solution is of the form $A_1 e^{\lambda_1 z} + A_2 e^{\lambda_2 z}$ for constants A_1 and A_2 that remain to be determined and λ_1, λ_2 that solve the characteristic equation above.

6. Now we're going to look for a *particular* solution to the full equation, including the forcing term, of the form $v(z) = Ke^{\alpha z}$. Characterize K. The full solution takes the form

$$v(z) = A_1 e^{\lambda_1 z} + A_2 e^{\lambda_2 z} + Ke^{\alpha z}$$

7. Consider now the case where the cost function (per unit of desired capital) takes the form:

$$C(\eta) = C_l 1_{\eta>0} + C_u 1_{\eta<0}$$

where $\eta = I/K^*$ and $1_{\eta>0}$ is the indicator function that takes the value 1 when $\eta > 0$ and 0 otherwise, $1_{\eta<0}$ is the indicator function that takes the value 1 when $\eta < 0$ and 0 otherwise. Define the trigger and target points U, u, L, l. Write down the equilibrium conditions that U, u, L, l must satisfy. Show that $u = l$ and interpret.

8. (Harder and not graded). Suppose that $\tilde{\mu} = \sigma^2/2 = 1$, $\alpha = 0.5$, $r = 0.1$, $C_u = C_l = 0.1$. Numerically solve for U, u, l and plot the value function $v(z)$ as well as Tobin's $q = v'(z)$ over the range $[U, L]$.

3 Autarky Interest Rates and the Current Account

Consider a small open economy in a world that lasts two periods, $t = 1$ and 2. There are 2 states of the world in period 2 ($s = 1, 2$) and we denote $\pi(s)$ the probability of state s being realized. This small economy is endowed with a quantity Y_1 of the single good in period 1 and a quantity $Y_2(s)$ in period 2.

The economy is populated by a representative agent with the following preferences:

$$U = \ln C_1 + \beta E[\ln(C_2)]$$

where $0 < \beta \leq 1$ is the discount factor.

Markets are complete: the representative household can buy/sell Arrow-Debreu securities that pay if and only if state s is realized in period 2. The price in period 1 of such securities is $q(s)$.

a. Express period 1 consumption C_1 as a function of home's total wealth W_1 (that you will define);

b. Similarly, express period 2 consumption $C_2(s)$ as a function of W_1, $q(s)$ and $\pi(s)$;

c. Assuming that the country starts in period 1 without any external debt, and that $\beta(1+r) = 1$, where r denotes the risk-free real interest rate, show that we can write the period 1 current account as:

$$CA_1 = \frac{\beta}{1+\beta}\left[Y_1 - \bar{Y}_2\right]$$

where $\bar{Y}_2 = \sum_s q(s) Y_2(s)(1+r)$. Interpret.

d. Derive an expression for the *autarky* interest rate, r^A. In the bond only economy, the current account is simply a function of $r - r^A$. Is it the case here?

e. Calculate now the real interest r^{CA} that would prevail if the representative household was prevented from engaging in *intertemporal* trade but was still *allowed to trade in state contigent claims*. Show that it satisfies:

$$\left(1 + r^{CA}\right)^{-1} = \beta\frac{Y_1}{\bar{Y}_2}$$

f. Rewrite the current account CA_1 as a function of $r - r^{CA}$. What do you conclude?

g. Characterize the demand $B_2(s)$ for the state-contigent asset that pays in state s and express it as a function of the price of the Arrow-Debreu security $q(s)$ and the price of the Arrow Debreu security $q^{CA}(s)$ that would obtain when intertemporal trade is prohibited (that is, when $r = r^{CA}$). What do you conclude about the pattern of gross capital flows? Interpret.