Econ 101A – Final Exam
We May 9, 2012.

You have 3 hours to answer the questions in the final exam. We will collect the exams at 2.30 sharp. Show your work, and good luck!

Problem 1. Utility Maximization. (45 points) Consider the following utility maximization problem:

$$\begin{aligned} \max_{x,y} u(x,y) &= \alpha \log(x) + (1-\alpha)\log(y) \\ \text{s.t. } p_x x + p_y y &= M, \end{aligned}$$

with $0 < \alpha < 1$. As usual, interpret p_x as the price of good x, p_y as the price of good y, and M as income.

1. Write down the Lagrangean function and derive the first order conditions for this problem with respect to x, y, and λ. (5 point)

2. Solve explicitly for x^* and y^* as a function of p_x, p_y, M,and α. (10 points)

3. How do these solutions for x^* and y^* compare to the solution for a standard Cobb-Douglas utility function $u(x,y) = x^\alpha y^{1-\alpha}$? Explain as clearly as you can the reasons for the relationship between the solutions for these two utility functions. (10 points)

4. Notice that the utility function is defined only for $x > 0, y > 0$. Does your solution for x^* and y^* satisfies these constraints? What assumptions you need to make about p_x, p_y and M so that $x^* > 0$ and $y^* > 0$? (5 points)

5. Now, consider the fact that the government needs to raise extra revenue to pay for an expensive war. It is considering two possibilities: an income tax t_s and an indirect tax t_i. The utility maximization problem becomes

$$\begin{aligned} \max_{x,y} u(x,y) &= \alpha \log(x) + (1-\alpha)\log(y) \\ \text{s.t. } p_x(1+t_i)x + p_y(1+t_i)y &= M(1-t_s). \end{aligned}$$

   Interpret the revised budget constraint (5 points)

6. Solve for the revised solutions for x^* and y^* and provide intuition for the effects of the added taxes on optimal consumption. (5 points)

7. Solve for the share of income s spent on good x, that is, in this case, $s = p_x(1+t_i)x^*/M(1-t_s)$. How is this share s affected by the two types of taxes? Interpret. (5 points)

Solution to Problem 1.

1. The Lagrangean is

$$L = \alpha \log(x) + (1-\alpha)\log(y) - \lambda(p_x x + p_y y - M)$$

   The first order conditions are:

$$x : \frac{\alpha}{x} - \lambda p_x = 0$$

$$y : \frac{1-\alpha}{y} - \lambda p_y = 0$$

$$\lambda : p_x x + p_y y - M = 0$$

2. From the first to foc's:

$$\frac{y}{x} = \frac{1-\alpha}{\alpha}\frac{p_x}{p_y}$$

   and with the third foc gives:

$$x^* = \alpha \frac{M}{p_x}$$

   and thus

$$y^* = (1-\alpha)\frac{M}{p_y}$$

3. These are the same demands as those derived with the Cobb-Douglas utility function. If we compose the Cobb-Douglas utility function with the increasing function $\log(\cdot)$, we get the objective we are maximizing in this problem. That is, denote with $v(x,y)$ the standard Cobb-Douglas function, then

$$\log v(x,y) = \log\left[x^{\alpha}y^{1-\alpha}\right] = \alpha\log(x) + (1-\alpha)\log(y) = u(x,y.)$$

   We know that preferences are maintained with monotonic increasing tranformations of any representative utility function (i.e. the resulting function also represents the same preferences). Thus, deriving the same demands is no surprise.

4. x^* and y^* satisfy the nonnegativity constraints iff $\alpha, 1-\alpha > 0$, with M, p_x and p_y also positive.

5. Indirect tax t_i increases the effective price of each good, by factoring up the price of each by the factor $(1+t_i)$. The tax on income, on the other hand, factors down income by $(1-t_s)$. The former is more indirect because it will not be borne if the consumers choose not to consume (which of course will not be the case).

6. The new first order conditions are:

$$x : \frac{\alpha}{x} - \lambda(1+t_i)p_x = 0$$

$$y : \frac{1-\alpha}{y} - \lambda(1+t_i)p_y = 0$$

$$\lambda : (1+t_i)p_x x + (1+t_i)p_y y - (1-t_s)M = 0$$

   giving,

$$\frac{y}{x} = \frac{1-\alpha}{\alpha}\frac{p_x}{p_y}$$

   as before, but with the new budget constraint:

$$x^* = \alpha\frac{(1-t_s)M}{(1+t_i)p_x}$$

$$y^* = (1-\alpha)\frac{(1-t_s)M}{(1+t_i)p_y}$$

   The direct tax decreases the total amount of income that can be spent on consumption, and thus thus factors down the numerator. The indirect tax effectively factors up the price, and thus factors up the denominator. We get the same Cobb-Douglas result that portion α of income is spent on good x, but now effective income and effective price of x factor in (with a similar intuition for y).

7.

$$
\begin{aligned}
s &= p_x (1+t_i)\, x^* / M (1-t_s) \\
&= \alpha
\end{aligned}
$$

The taxes do not change the share: 1) both prices are multiplied by $(1+t_i)$ and thus the effective price ratio is the same, and 2) preferences are not affected by taxes.

8. **Problem 2. Shorter Questions.** (55 points)

Problem 2a (*Expected Utility.*) Consider an individual with utility function $u(c)$ where c is consumption, with u which satisfies $u'(c) > 0$ for all c. The individual has a job with uncertain income, she earns w_H with probability p and $w_L < w_H$ with probability $1-p$. The agents has no other income and hence consumes $c = w$.

1. Write the expected utility of the individual. (5 points)

2. The government is considering providing an insurance policy which would provide a flat wage W which is constant, rather than fluctuating across states of the world, as an alternative to getting the uncertain market wage defined above. The government would like to know what is the level of sure pay W which would make the person *indifferent* between taking this insurance policy (and hence earning $\bar{W}$ for sure) and not taking it (and hence earning expected utility as in the point above). Write down the equation which implicitly defines this level $\bar{W}$. (5 points)

3. The government wants to know whether the insurance level $\bar{W}$ which makes the person indifferent is smaller than the expected wage $pw_H + (1-p)w_L$. As a consultant to the government, provide conditions under which it is indeed smaller. If you can, provide necessary and sufficient conditions. Explain as clearly as you can. (15 points)

Problem 2b (*Moral Hazard*). Consider the moral hazard (hidden action) problem which we considered in lecture. An agent is offered a contract $w = a + by$, where w is the wage, and y is the output, with $y = e + \varepsilon$, where e is (unobservable) effort and ε is noise. Remember that the agent has exponential utility which leads to the expected utility of a contract being equal to

$$EU(w) = a + be - \frac{\gamma}{2}b^2\sigma^2 - c(e)$$

4. Interpret the three parts in this expression (5 points)

5. Remember that effort is costly with $c(e)$ cost of effort which satisfies $c'(e) > 0$ and $c''(e) > 0$ for all e. Without solving for the overall problem, explain intuitively, but also as clearly as you can, why the firm will not set $b^* = 0$ in the optimum. What effort would workers choose for $b = 0$? (10 points)

6. Without solving for the overall problem, explain intuitively why the firm will set $b^* < b_{FB}$ in the optimum, where b_{FB} is the level of incentives which would achieve the optimal (i.e., first-best) level of production. (Remember that the optimal level of production is the one achieved when e is observable) (5 points)

7. In light of this, explain qualitatively what it means that the firm is facing a risk-incentive trade-off in setting the optimal piece rate b^*. (10 points)

Solution to Problem 2.

1. The expected utility is

$$pu\left(w_H\right)+\left(1-p\right)u\left(w_L\right)$$

2. The level $\bar{W}$ is defined by

$$u\left(\bar{W}\right)=pu\left(w_H\right)+\left(1-p\right)u\left(w_L\right)$$

or

$$\bar{W}=u^{-1}\left[pu\left(w_H\right)+\left(1-p\right)u\left(w_L\right)\right].$$

This term is sometimes called the certainty equivalent.

3. A sufficient condition is $u''\left(c\right)<0$ for all c. Necessary conditions are $u''\left(c\right)\not>0$ for all c nor can $u\left(c\right)$ be linear. We do not have necessary and sufficient conditions: we can have a $u\left(c\right)$ that is convex over some c but concave over most other c, while maintaining $\bar{W}<pw_H+\left(1-p\right)w_L$.

4. $a+be$ gives the expected wage from the contract. Because the agent is risk averse, $\frac{\gamma}{2}b^2\sigma^2$ is deducted due to the uncertainty in output. The b term measures the agent's exposure to the noise in output, as specified by the contract. In all, $\frac{\gamma}{2}b^2\sigma^2$ is increase in this exposure term, as well as the variance of the noice σ^2, and the agent's coefficient of absolute risk aversion, γ.

5. If $b=0$, the agent is not exposed to any noise, and because effort is costly, he will choose $e=0$. By exposing the agent to noise in output ($b>0$), the principle incentivises effort.

6. There is a problem with this question, so it was graded as extra credit. When effort is observable, the principal will completely internalize the risk preferences of the agent and provide full insurance, but will at the same time incentivize the agent to exert the optimal effort. Formally, the first best contract will be to offer $w=c(e_{FB})$ if e_{FB} is observed, and $-\infty$ or some other large punishment for any effort which violates this condition. So, b is not defined in the first best because the contract will not be linear in effort e. In the second best solution, the principle has to expose the agent to output noise ($b_{FB}>0$) to incentivise effort (though $e^*<e_{FB}$).

7. The firm does want to insure the risk averse agent, as this allows the firm to pay a smaller certain wage (a). However, the firm also needs to expose the agents in order to incentivise effort. Thus, it will not choose to entirely internalize the agent's risk aversion, but it will partially due to this risk-incentive trade-off.

Problem 3. Static Games and Dynamic Games. (60 points).

1. State the definition of a Nash equilibrium for a game in which there are 2 players $i = 1$ and $i = 2$, each of which with utility function $u_i(s_i, s_{-i})$, as a function of strategies s_i and s_{-i}. (5 points)

2. Consider the following 2×2 game played simultaneously:

		P2	
		L	R
P1	T	1 , 10	1 , 1
	B	2 , α	0 , 1

   First, assume $\alpha = 2$ for this and the next points, until stated otherwise. Find the pure Nash equilibrium (if any). (That is, do not worry yet about equilibria in mixed strategies. (5 points)

3. Now, consider the dynamic game in which player 1 moves before player 2. The payoffs remain the same, draw the decision tree for this dynamic game. Write down the set of possible strategies for each player. (10 points)

4. Solve for the sub-game perfect equilibrium *strategies* and also compute the equilibrium payoff. Is there an advantage to moving first in this game (versus moving simultaneously)? How about to moving second (versus moving simultaneously)? (5 points)

5. In light of the solution of the previous point, comment on this – is this right or wrong, and why: '*Having player 1 move first advantages player 2 because Player 2 threatens to go R if player 1 goes B, and to go L if player 1 goes R. Facing this threat, player 1 goes T, because he prefers a payoff of 1 (from T,L) to a payoff of 0 (from B,R). The threat works because in equilibrium player 1 goes T, and hence never finds out whether player 2 would actually play R if player 1 were to go B. Hence, player 2 can get 10 in equilibrium*' (10 points)

6. Now consider the original static game, but from now on assume $\alpha = 0$. Find all the Nash equilibria of this game, including the mixed strategies. Denote by p the probability player 1 plays T, and by q the probability that player 2 plays L. Compute and graph the best response functions, and show where they intersect, which are the Nash equilibria (or equilibrium if there is only one). (10 points)

7. Compute the expected payoffs for each player in the Nash equilibrium above. (5 points)

8. Now, let player 1 move first again in a dynamic game. Draw the tree and solve for the sub-game perfect equilibrium *strategies* and also compute the equilibrium payoff. (5 points)

9. Is there an advantage to moving first in this game (versus moving simultaneously)? How about to moving second (versus moving simultaneously)? (5 points)

Solution to Problem 3.

1. A strategy profile s^* is a Nash equilibrium iff $\forall i$, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$ for every s_i.

2. By underlining the best response, it is clear that player 2's best response is always L. Player 1's is to play B if the other player plays L and to play T if the other player plays R. Hence, the only Nash equilibrium is $(B.L)$

3. Player 1's strategies are just T or B, as he only plays in the initial node. Player 2 has strategies: $(L|T, L|B)$, $(L|T, R|B)$, $(R|T, L|B)$, or $(R|T, R|B)$, where the first corresponds to the strategy corresponding to player 1 moving T, and the second to the strategy corresponding to player 1 moving B.

4. We solve by backwards induction. The second player's optimal strategy is to play left in both nodes. or $(L|T, L|B)$. Expecting this, player 1 plays B. The equilibrium payoffs are (2,2). There is no advantage here to moving first or second compared to moving simultaneously, as the equilibrium is the same.

5. The threat is not sub-game perfect. Player 1 knows that, if he goes B, it will be in the interest of Player 2 to go L. Hence, Player 1 calls the bluff of player 1, goes B, and in response Player 2 goes L (not R as in the threat). The type of empty threats as in the wording above is precisely what sub-game perfect equilibrium eliminates.

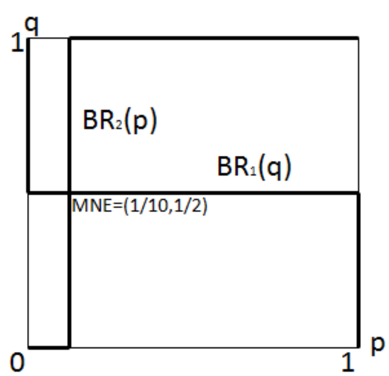
6. We need to find the mixed Nash. Player 1's EU from playing T: $q + (1-q) = 1$, player 1's EU from playing B: $2q + 0(1-q) = 2q$. Hence, it is a best response for player 1 to play T if $1 \geq 2q$, or $q \leq 1/2$. For $q^* = \frac{1}{2}$ player 1 is indifferent. Now let's do player 2. Player 2's EU from playing L: $10p + 0(1-p) = 10p$ and player 2's EU from playing B: $1p + 1(1-p) = 1$. Thus it is a best response for player 2 to play L if $10p \geq 1$ or $p \leq 1/10$. For $p^* = \frac{1}{10}$ player 2 is indifferent:

$$BR_1(q) = \begin{cases} 0 & \text{if } q \geq 1/2 \\ \text{any } p \in [0,1] & \text{if } q = 1/2 \\ 1 & \text{if } q \leq 1/2 \end{cases}$$

and

$$BR_2(p) = \begin{cases} 0 & \text{if } p \leq 1/10 \\ \text{any } q \in [0,1] & \text{if } p = 1/10 \\ 1 & \text{if } p \geq 1/10 \end{cases}$$

The following graph provides the plot, the only Nash equilibrium is $p^* = 1/10$ and $q^* = 1/2$.



7. To compute the expected payoffs for player 1, notice that in equilibrium he is indifferent between going T and going B (since he is playing a mixed strategy). Hence, let's compute the expected payoff from going T, which is 1 – you'd find the same for going B. The expected payoff of player 2, who is also indifferent, can be computed as the payoff of going R, which is 1.

8. If player 1 moves first, by backward induction the only sub-game perfect equilibria are for player 2 to go L if player 1 goes T and to go R if player 1 goes B. Anticipating this, player 1 prefers to go T which guarantees a payoff of 1 rather than going B which gives a payoff of 0. Hence, the SPE strategies are $1- > T$, $2- > (L|T, R|B)$. The equilibrium payoff is 1 for player 1 and 10 for player 2.

9. Player 1 earns the same expected payoff in the dynamic game than in the static game, but player 2 earns much more in the dynamic game, even though he is the second mover (10 compared to 1)! Hence, it is not always bad to be a second mover.

Problem 4. (Production) (65 points)

International Toys L.L.C. produces action figures for the Japanese and American markets. For one particular doll, the Auto-Action-Rambo-Teletubby doll (that's one doll, it's both lethal and huggable at the same time, a Matt Leister unique design), International Toys L.L.C. sells exclusively to Toy-Japan in Japan and Toys-R-Us in America. Toy-Japan's demand function is $D_J(p_J) = \frac{40-2p_J}{3}$ (giving inverse demand $p_J(y_J) = 20-\frac{3}{2}y_J$), and Toys-R-Us's demand is $D_A(p_A) = 16-p_A$ (giving inverse demand $p_A(y_A) = 16-y_A$), where quantity is measured in thousands of dolls. International Toys faces a cost function $C(Y) = 4 + Y^2$, where Y is the total production of Auto-Action-Rambo-Teletubby dolls in thousands of units.

1. Assume first that International Toys L.L.C. sells in a perfectly competitive market and faces price p_{PC}. Derive the marginal and average cost and plot them in a graph with Y on the x axis. (5 points)

2. Derive the supply function of Auto-Action-Rambo-Teletubby dolls $Y^S(p_{PC})$. Use the graph if it helps, but make sure to write down the supply function analytically. (10 points)

3. Suppose now that International Toys L.L.C. acts as a monopolist in both countries, and can choose separate prices p_J and p_A to charge the two chain stores (third-degree price discrimination). Write down the profit function from selling in both countries as function of the quantities sold Y_J and Y_A. [Remember that the costs are given by $C(Y) = 4 + (Y_J + Y_A)$] (5 points)

4. Solve the maximization problem and determine the optimal quantities sold Y_J^* and Y_A^*. Using this, determine also what prices p_J and p_A the monopolist will set. (10 points)

5. Now assume that a Japanese-American law does not allow separate pricing between the two countries, so $p_J = p_A = p$. Derive the aggregate demand $D(p) = D_J(p) + D_A(p)$ which International Toys faces. [Note: Here keep in mind that demand in each country cannot be negative, that is, $D_J(p) \geq 0$ and $D_A(p) \geq 0$, it may help to draw the demand curves and add them] (10 points)

6. Using the aggregate demand function, solve for the profit maximizing price p that International Toys will charge. [Hint: Here you can assume $p < 16$] Here it may be easier to write the profit maximization of the monopolist as maximization with respect to the price p. Find the implied sales in each country (that is, y_J and y_A). (5 points)

7. Continue assuming that International Toys L.L.C. charges one monopoly price in the two countries. Now, however, a Chinese company makes an identical toy which is also sold in America. Since consumers are indifferent between the Chinese version and the International Toys version, the price in America will drop to the price of production of the Chinese toy, which is 11. International Toys L.L.C. now has two strategies: (i) to keep selling in both markets at $p = 11$; (ii) to sell only in Japan at the profit-maximizing price. Compute the profits for each of the two possibilities and hence the overall optimum. Does the firm continue to sell in America? (10 points)

8. Has the consumer surplus increased or decreased in America as a function of the entry of the Chinese toy? [You should not need to solve for the consumer surplus to answer the question] How about in Japan? Explain. (10 points)

Solution to Problem 4.

1. If International Toys L.L.C. were to set one price in a perfectly competitive market, we can derive the supply function of Auto-Action-Rambo-Teletubby dolls as follows. The marginal cost is: $C'(Y) = 2Y$, with average cost: $C'(y) = \frac{4}{Y} + Y$.

2. The marginal cost is above average cost when $2Y \geq \frac{4}{Y} + Y \Leftrightarrow Y \geq 2$, or a price of 4. Thus, the supply function is:

$$Y^S(p) = \begin{cases} 0 & \text{if } p < 4 \\ \frac{p}{2} & \text{if } p \geq 4 \end{cases}$$

3. International Toys L.L.C. maximizes profits defined as the sum of the revenue in the two markets, minus the costs

$$\max_{y_J, y_A} \left(20 - \frac{3}{2} y_J\right) y_J + (16 - y_A)\, y_A - \left(4 + (y_J + y_A)^2\right)$$

4. F.O.C's give:

$$y_J : 20 - 3y_J - 2\,(y_J + y_A) = 0$$

$$y_A : 16 - 2y_A - 2\,(y_J + y_A) = 0$$

giving:

$$y_A = 10 - \frac{5}{2} y_J$$

$$\begin{aligned} 16 - 2\left(10 - \frac{5}{2} y_J\right) - 2\left(y_J + 10 - \frac{5}{2} y_J\right) &= 0 \\ y_J &= \frac{24}{8} = 3 \end{aligned}$$

$$y_A = 10 - \frac{5}{2} 3 = \frac{5}{2}$$

$$\Longrightarrow p_J = 20 - \frac{3}{2} 3 = \frac{31}{2} = 15.5$$

$$p_A = 16 - \frac{5}{2} = \frac{27}{2} = 13.5$$

5. The aggregate demand will be

$$D(p) = \begin{cases} 0 & \text{if } p \geq 20 \\ \frac{40-2p}{3} & \text{if } p \in [16, 20] \\ \frac{88}{3} - \frac{5}{3} p & \text{if } p \in [0, 16] \end{cases}$$

giving inverse demand.

6. Seeing that both prices above lie below 16, the optimal p will also lie below 16 (it should be in between 13.5 and 15.5), so that both markets are included:

$$\max_p \left(\frac{88}{3} - \frac{5}{3} p\right) p - \left(4 + \left(\frac{88}{3} - \frac{5}{3} p\right)^2\right)$$

with f.o.c.:

$$\begin{aligned} \frac{88}{3} - \frac{10}{3} p + 2\frac{5}{3}\left(\frac{88}{3} - \frac{5}{3} p\right) &= 0 \\ \frac{88}{10} - p + \left(\frac{88}{3} - \frac{5}{3} p\right) &= 0 \\ p &= \frac{3}{8} \frac{13 * 88}{30} = \frac{13 * 11}{10} = 14.3 \end{aligned}$$

giving total production $Y = \frac{88}{3} - \frac{5}{3} \frac{13*11}{10} = \frac{11}{2}$, or in each country, $D_J(\frac{13*11}{10}) = \frac{40 - 2\frac{13*11}{10}}{3} = \frac{57}{15} = 3.8$ and $D_A(\frac{13*11}{10}) = 16 - \frac{13*11}{10} = \frac{17}{10} = 1.7$.

7. International Toys L.L.C. can either sell at price 11 in both countries supplying $\frac{11}{2}$ (from part 1), or leave the American market and exclusively sell to Toy-Japan. At price 11 International Toys L.L.C. will realize profits:

$$\frac{11^2}{2} - \left(4 + \left(\frac{11}{2}\right)^2\right) = \frac{105}{4} = 26.25$$

If it leaves the American market. Selling only to Japan, it realizes profits:

$$\max_{y_J} \left(20 - \frac{3}{2} y_J\right) y_J - \left(4 + y_J^2\right)$$

with f.o.c.:

$$\begin{aligned} 20 - 3y_J - 2y_J &= 0 \\ y_J &= 4 \\ p_J &= 20 - \frac{3}{2}4 = 14 \end{aligned}$$

That is, it realizes profits

$$(14)\,4 - \left(4 + 4^2\right) = 36$$

Thus, it leaves the American market.

8. Because the price in America has dropped, the consumer surplus has increased. On the other hand, prices has increased in Japan (International Toys L.L.C. act as a monopolist in Japan), Japanese consumer surplus has decreased. We would have to calculate these changes to determine if total consumer surplus has increased or decreased.