Econ 101A – Final exam
May 14, 2013.

Do not turn the page until instructed to.
Do not forget to write Problems 1 in the first Blue Book and Problems 2, 3 and 4 in the second Blue Book.

Econ 101A – Final Exam
May 14, 2013.

Do not forget to write Problems 1 in the first Blue Book and Problems 2, 3 and 4 in the second Blue Book. Good luck on solving the problems!

Problem 1. Second-price auction (85 points) Two individuals participate in a sealed bid auction. Individual 1 values the good v_1 and individual 2 values it v_2. Assume $v_1 > v_2$. Assume that both individuals know their own value, as well as the value for the opponent. The two individuals write their valuation inside a sealed envelope and submit it to the auctioneer simultaneously (that is, they cannot observe the other bidder's bid). The auctioneer assigns the good to the individual with the highest bid at the *second* highest price. (In the case of tied bids, the auctioneer assigns the good with probability 0.5 to the first player) Remember, this is to a first approximation the auction format used by eBay. Formally, denote by b_i the bid of individual i. The payoff function of individual i is

$$u_i(b_i) = \begin{cases} v_i - b_{-i} & \text{if} \quad b_i > b_{-i} \\ (v_i - b_{-i})/2 & \text{if} \quad b_i = b_{-i} \\ 0 & \text{if} \quad b_i < b_{-i} \end{cases} \tag{1}$$

where b_{-i} is the bid of the other player.

1. Explain why the payoff function is given by the expression (1). (5 points)
2. Give the general definition of an equilibrium in dominant strategies and of Nash Equilibrium. (10 points)
3. Show that, no matter what player $-i$ bids, it is always (weakly) optimal for player i to bid v_i, that is, $(b_1^*, b_2^*) = (v_1, v_2)$ is an equilibrium in dominant strategies (20 points)
4. Using the definitions in (1), conclude that $(b_1^*, b_2^*) = (v_1, v_2)$ is a Nash Equilibrium. (10 points)
5. Assume now that players play sequentially. Player 1 bids b_1 first, and player 2 bids next after observing the bid b_1. How do we find the sub-game perfect equilibria in this case? (10 points)
6. Is it a sub-game perfect equilibrium that player 1 bids $b_1^* = v_1$ and player 2 bids $b_2^* = v_2$, no matter what the history? Go through all the steps of the argument. (15 points)
7. Consider now a first-price simultaneous sealed bid auction with two players, that is, the top bidder pays his bid. The payoff function of player i is

   $$u_i(b_i) = \begin{cases} v_i - b_i & \text{if} \quad b_i > b_{-i} \\ (v_i - b_i)/2 & \text{if} \quad b_i = b_{-i} \\ 0 & \text{if} \quad b_i < b_{-i}. \end{cases}$$

   Show that $(b_1^*, b_2^*) = (v_1, v_2)$ is NOT a Nash Equilibrium for the case $v_1 \neq v_2$. [Do not attempt to characterize the Nash equilibria, just show that something is not a Nash equilibrium] (15 points)

Solution to Problem 1.

1. Bidder i does not get the object if $b_i < b_{-i}$, hence the 0 payoff in that case. For $b_i > b_{-i}$, the player gets the object (with valuation v_i) and pays the second highest bid, which is b_{-i}, hence the payoff. If $b_i = b_{-i}$, the person gets the object with probability .5; using expected utility, the valuation is $.5 * 0 + .5 * (v_i - b_{-i})$.

2. An equilibrium $\left(s_i^*, s_{-i}^*\right)$ in dominant strategies satisfies

$$u_i\left(s_i^*, s_{-i}\right) \geq u_i\left(s_i, s_{-i}\right) \text{ for all } s_i, \text{ for all } s_{-i}, \text{ and for all players } i.$$

   A Nash equilibrium $\left(s_i^*, s_{-i}^*\right)$ satisfies

$$u_i\left(s_i^*, s_{-i}^*\right) \geq u_i\left(s_i, s_{-i}^*\right) \text{ for all } s_i \text{ and for all players } i.$$

   The difference is that in the Nash equilibrium the comparison of payoffs for player i is done holding constant the other players at their optimal strategy s_{-i}^*, not for all possible strategies s_{-i}.

3. It is enough to consider three cases. If $b_{-i} > v_i$, player i can either win the item with positive probability by bidding $b_i \geq b_{-i}$, which brings her negative utility since $v_i - b_{-i} < 0$, or bid less than player $-i$ which brings her zero utility. The best outcome is clearly the latter one, which is achieved by bidding i's true valuation (among other possible bids). If $b_{-i} < v_i$, player i can either win the item for sure by bidding $b_i > b_{-i}$ for utility $v_i - b_{-i} > 0$, win it with probability 1/2 by bidding $b_i = b_{-i}$ for utility $\frac{v_i - b_{-i}}{2} < v_i - b_{-i}$, or not win it by bidding $b_i < b_{-i}$ for zero utility. The best outcome is the first one, which can again be achieved by bidding i's true valuation. Finally, if $b_{-i} = v_i$, it is not hard to check that both winning and not winning the item brings utility of zero to i so any bid (including $b_i = v_i$) is a best response. In all three cases, i achieves her best outcome by bidding her true valuation and therefore that is a weakly dominant strategy for her.

4. From point (2), one can see that if $\left(s_i^*, s_{-i}^*\right)$ is an equilibrium in dominant strategies, it must also be a Nash equilibrium. The reason is that the definition of dominant strategies is more restrictive, it requires that s_i^* be the best choice for any possible choice of strategies of the other players s_{-i}. If that is satisfied, *a fortiori* it will be the case that it holds holding the other players at s_{-i}.

5. If the players play sequentially, we find the subgame perfect equilibria by backward induction. That is, we start from the last period, and consider the optimal strategies for, in this case, player 2 as a function of the history. Then we work backwards and determine the optimal strategy in the earlier period, in this case by player 1, taking into account the optimal strategies derived for the last period.

6. Let's start from the last period. Is it indeed optimal (at least weakly) for player 2 to bid her value b_2, for all the histories? Before, we showed that it is (weakly) a best response for player 2 to bid $b_2^* = v_2$, no matter what b_1 is. Hence, it is optimal for player 2 to bid $b_2^* = v_2$ *in all histories.* Then, move back to period 1. Player 1 knows that no matter what she bids, player 2 will bid v_2 in the second player. We already showed that it is optimal for player 1 to bid v_1 no matter what player 2 bids. Hence, $b_1^* = v_1$ satisfies the sub-game perfection as well.

7. In a first-price auction, the highest bidder pays their own bid, hence the difference from the above payoffs. Intuitively, once a winning bidder pays their own bid and we are not any more in a second price auction setting, the bidder will have an incentive to shade their value and not bid their valuation. We now show formally that indeed $(b_1^*, b_2^*) = (v_1, v_2)$ is NOT a Nash Equilibrium for the case $v_1 \neq v_2$. Suppose $v_1 > v_2$ without loss of generality (otherwise, swap the argument). Notice that in this supposed equilibrium, the payoff of player 1 is 0 since $v_1 - b_1^* = v_1 - v_1 = 0$. (That is, bidder 1 wins the auction, but gets no surplus because he is pay as bid her full value). By deviating to any bid $\hat{b}_1$ with $v_2 < \hat{b}_1 < v_1$, player 1 increases his payoff since he still wins the auction but now gets positive payoff $v_1 - \hat{b}_1 > 0$. Since there is a profitable deviation by at least one of the parties, this is a not a Nash Equilibrium. In fact, but you definitely were *not* supposed to derive this, there is no equilibrium here in pure strategies, only in mixed strategies.

Problem 2. Short questions. (70 points)

1. **Cost curves.** For each of the following cost functions, plot as well as derive analytically the marginal cost $c'(y)$, the average cost $c(y)/y$ and the supply function $y^*(p)$:

   (a) $c(y) = 10y^2$ (10 points)

   (b) $c(y) = 10 + 10y^2$ (10 points)

   (c) $c(y) = 5y$ (10 points)

   (d) For each of the cost functions above, say what the returns to scale are for the underlying technology and whether there is a fixed cost. (10 points)

2. **Game Theory.** Compute the pure-strategy and mixed strategy equilibria of the following coordination game. Call u the probability that player 1 plays Up, $1-u$ the probability that player 1 plays Down, l the probability that Player 2 plays Left, and $1-l$ the probability that Player 2 plays Right. (30 points)

1\2	Left	Right
Up	3, 3	1, 1
Down	1, 1	2, 2

Solution to Problem 1.

1. We proceed case-by-case:

   (a) $c'(y) = 20y$, $c(y)/y = \frac{10y^2}{y} = 10y$. The marginal cost function is always (weakly) above the average cost function. Supply function:

$$S(p) = \frac{1}{20}p \text{ if } p > 0$$

   Decreasing returns to scale and no fixed cost.

   (b) $c'(y) = 20y$, $C(y)/y = \frac{10}{y} + 10y$. Supply function:

$$S(p) = \begin{cases} \frac{1}{20}p & \text{if} \quad p \geq 20 \\ 0 & \text{if} \quad p < 20 \end{cases}$$

   Decreasing returns to scale and fixed cost of 10.

   (c) $c'(y) = c(y)/y = 5$. Supply function:

$$S(p) = \begin{cases} \to +\infty & \text{if} \quad p > 5 \\ \text{any } q \in [0, \infty) & \text{if} \quad p = 5 \\ 0 & \text{if} \quad p < 5 \end{cases}$$

   Constant returns to scale and no fixed cost.

2. The pure strategy Nash equilibria can be found in the matrix once we underline the best responses for each player:

1\2	Left	Right
Up	$\underline{3},\underline{3}$	$1,1$
Down	$1,1$	$\underline{2},\underline{2}$

The equilibria therefore are $(s_1^*, s_2^*) = (U, L)$ and $(s_1^*, s_2^*) = (D, R)$. To find the mixed strategy equilibria, we compute for each player the expected utility as a function of what the other player does. We start with player 1. Player 1 prefers Up to Down if

$$lu_1(U,L) + (1-l)\,u_1(U,R) \geq lu_1(D,L) + (1-l)\,u_1(D,R)$$

or

$$3l + (1-l) \geq l + 2(1-l)$$

or

$$l \geq 1/3.$$

Therefore, the Best Response correspondence for player 1 is

$$BR_1^*(l) = \begin{cases} u = 1 & \text{if} \quad l > 1/3; \\ \text{any } u \in [0,1] & \text{if} \quad l = 1/3; \\ u = 0 & \text{if} \quad l < 1/3. \end{cases}$$

Since the game is symmetric, it is not hard to see that the Best Response correspondence for player 2 then is

$$BR_2^*(u) = \begin{cases} l = 1 & \text{if} \quad u > 1/3; \\ \text{any } l \in [0,1] & \text{if} \quad u = 1/3; \\ l = 0 & \text{if} \quad u < 1/3. \end{cases}$$

Plotting the two Best Response correspondences, we see that the three points that are on the Best Response correspondences of both players are $(\sigma_1^*, \sigma_2^*) = (u = 1, l = 1)$, $(u = 0, l = 0)$, and $(u = 1/3, l = 1/3)$. The first two are the pure-strategy equilibria we had identified before, the other one is the additional equilibrium in mixed strategies.

Problem 3. Monopoly and Duopoly. (90 points). Initially there is one firm in a market for cars. The firm has a linear cost function: $C(q) = q$. The market inverse demand function is given by $P(Q) = 10 - Q$.

1. What price will the monopolist firm charge? What quantity of cars will the firm sell? (10 points)
2. Plot the solution graphically: derive and plot the marginal cost, the marginal revenue, and the demand curve, and find the solution. (10 points)
3. How much profit will the firm make? (5 points)
4. Now, a second firm enters the market. The second firm has an identical cost function. What will the Cournot equilibrium output for each firm be? Compare the quantity produced and the price to the monopoly case (10 points)
5. Similarly to what you did in point (2) derive the Cournot solution graphically by plotting the best response curves in a graph with q_1 in the x axis and q_2 in the y axis – how do you find the equilibrium? (10 points)
6. What is the Stackelberg equilibrium output for each firm if firm 2 enters second? (10 points)
7. How much profit will each firm make in the Cournot game? How much in Stackelberg? (10 points)
8. Using a graphical plot of the demand function, derive a formula for the consumer surplus as the area below the demand curve and above the equilibrium price. Derive it for the cases of monopoly, Cournot and Stackelberg. Which yields the higher consumer surplus? (15 points)
9. Debate this assertion in light of your answer at the point above: 'Market power generates consumer losses because it leads to a higher market price. In particular, the welfare losses are linearly increasing in the market price.' (10 points)

Solution to Problem 2.

1. The first two questions are about the case of monopoly. A monopolistic firm maximizes profits:

$$\max_{y} (10 - q) q - q.$$

and the first order condition is

$$10 - 2q_M^* - 1 = 0.$$

It follows that

$$q_M^* = \frac{10 - 1}{2} = 9/2$$

and

$$p_M^* = 10 - q_M^* = 10 - 9/2 = 11/2.$$

2. See plot, the equilibrium quantity is at the crossing of the marginal cost curve $MC = 1$ and the marginal revenue curve, $MR = 10 - 2q$, then you find the price on the demand curve.
3. As for the profits in monopoly, they equal

$$\pi_M^* = (10 - q_M^*) q_M^* - q_M^* = (10 - 9/2) 9/2 - 9/2 = 81/4$$

4. We are now in a Cournot competition set-up. A duopolistic firm maximizes profits:

$$\max_{y_i} (10 - (q_i + q_{-i})) q_i - q_i. \quad (2)$$

The first order condition of problem (2) is

$$10 - 2q_i^* - q_{-i}^* - 1 = 0.$$

It follows that the first order conditions for the two firms are

$$10 - 2q_1^* - q_2^* - 1 = 0.$$

and

$$10 - 2q_2^* - q_1^* - 1 = 0.$$

From the first order condition, we get $q_2^* = 9 - 2q_1^*$. We substitute this expression into the second first order condition to get $9 - 2(9 - 2q_1^*) - q_1^* = 0$ or $3q_1^* = 9$, so

$$q_1^* = 3$$

and

$$q_2^* = 9 - 2q_1^* = 9 - 2 * 3 = 3.$$

Not surprisingly, the quantities produced by firms 1 and 2 are equal. The total market output $Q^* = 2 * 3 = 6$, larger than in monopoly. The price is $P(Q^* = 6) = 10 - 6 = 4$ and therefore lower than in monopoly.

5. See the figure, the equilibrium is where the two best response functions cross.

6. In the Stackelberg case, firm 1 maximizes taking into account the reaction function of firm 2. The reaction function of firm 2 is $q_2^*(q_1) = (9 - q_1)/2$. Therefore, firm 1 maximizes

$$\max_{q_1} (10 - (q_1 + 9/2 - q_1/2))\, q_1 - q_1 \text{ or}$$
$$\max_{q_1} (11/2 - q_1/2)\, q_1 - q_1$$

which leads to the first order conditions

$$11/2 - q_1 - 1 = 0$$

or

$$q_1^* = 9/2$$

and, using the reaction function of firm 2,

$$q_2^* = (9 - q_1^*)/2 = 9/4.$$

So the first mover produces twice as much. The total market output is $Q^* = 9/2 + 9/4 = 27/4$, larger than in Cournot.

7. In the Cournot case, the profit for either firm is

$$\pi_C = (10 - 6)\, 3 - 3 = 9.$$

Profits in Stackelberg differ. The profit of the leader is

$$\pi_S^1 = (10 - 27/4)\, 9/2 - 9/2 = 9/4 * 9/2 = 81/8 > \pi_C,$$

while the profit of the follower is

$$\pi_S^2 = (10 - 27/4)\, 9/4 - 9/4 = 9/4 * 9/4 = 81/16 < \pi_C$$

8. The consumer surplus is the area below the demand curve and above the price. The base of the triangle will be Q^*, with the height being $10 - P^*$. Hence, the consumer surplus is given by $Q^* * (10 - P^*)/2 = (10 - P^*)^2/2$, where we used the fact that $Q = 10 - P$ by the demand curve. Hence, the consumer surplus is maximized for the lowest P. Consumers prefer the Stackelberg duopoly, which has the highest total production.

9. The first part of the assertion is certainly correct. Market power, which is greatest in the monopoly case but also present in the duopoly case for Cournot or Bertrand, leads to a higher price and a lower quantity sold. Since the consumer surplus is $(10 - P^*)^2/2$, it is clear that a higher price lowers the surplus. The second part of the assertion, though, is incorrect. The surplus is a *quadratic,* not linear, function of the price. Increasing the price leads to losses in surplus that are more than linear. There is a parallel here to the deadweight loss discussion we had in class.

Problem 4. Worker Effort and Altruism. (90 points) Consider the case of a firm with employees. The employee chooses the effort e. The firm pays the employees both a salary W and a piece-rate w times the units of effort e. Unlike in the moral hazard case we dealt with in class, in this case the effort is observable by the firm. The firm earns revenue pe for every unit e produced by the worker. The timing is such that the firm first sets the pay package (W, w) and then the worker chooses the optimal effort e. The worker has linear utility over money (that is, is risk-neutral) and thus gains utility $W + we$ from earnings at the workplace. In addition, the worker pays a cost of effort $ce^2/2$ (with $c > 0$) from exerting effort at the workplace. Hence the worker maximizes

$$\max_e W + we - ce^2/2$$

1. Solve for the optimal e^* of the worker as a function of W, w, and c. (10 points)

2. How does the optimal effort of the worker depend on the piece rate w? How about on the flat pay W? How about on the cost parameter c? Discuss the intuition. (10 points)

3. Consider now the problem of the firm. Remember that the firm revenue is pe. Thus the firm seeks to maximize

$$\begin{aligned} & \max_{W,w} pe - W - we \\ s.t. e^* &= e^* (W, w) \end{aligned}$$

   The firm chooses optimally the piece rate w and the flat-pay W subject to $w \geq 0$ and $W \geq 0$. Plug into the maximization problem the e^* you derived above. First, use a qualitative argument to find the optimal W^*. Then take first order conditions with respect to w. What is the solution? [If you have trouble here, just skip to the next point] (10 points)

4. Let's go back now to the employee problem and consider the case in which the worker is altruistic towards the employer by putting a weight α on the profits of the firm, with $0 < \alpha < 1$. In this case, the worker solves the problem

$$\max_e W + we - ce^2/2 + \alpha [pe - W - we] .$$

   Derive the optimal effort e^*_α in this case. (10 points)

5. Now we compare the optimal effort with altruism e^*_α derived in point (4) for $0 < \alpha < 1$ to the optimal effort without altruism e^* derived in point (1). In doing the comparison, hold constant c, p, and also w, and W. Also, assume $p > w$. Which is larger, e^*_α or e^*? (Remember $0 < \alpha < 1$) Discuss the intuition. (10 points)

6. How do the two efforts e^* and e^*_α respond to an increase in the value of the product p? Compare the derivatives of e^* and e^*_α with respect to p . Discuss the intuition. (10 points)

7. How do the two efforts e^* and e^*_α respond to an increase in the value of the piece-rate w? Compare the derivative of e^* and e^*_α with respect to w and discuss which effort is more sensitive to the piece rate. Discuss the intuition. (10 points)

8. Consider now the case of a social planner who aims to maximize overall welfare in the economy. This particular social planner wants to maximize the sum of the utility of the employee and of the firm. Therefore, this planner maximizes

$$\left(W + we - ce^2/2\right) + (pe - W - we) .$$

   Find the solution for e^*_{SP} which maximizes this expression. (10 points)

9. For which value of altruism α does the solution e^*_α equate the one set by the social planner? Explain the intuition. (10 points)

Solution to Problem 4.

1. The f.o.c.s are
$$w - ce^* = 0$$
and thus
$$e^* = w/c.$$
The second order condition is
$$-c < 0$$
which is satisfied since $c > 0$. To get full credit, you should have remembered to check the second order conditions.

2. An increase in piece rate w increases the incentive to exert effort and hence leads to a higher effort. Instead, an increase in the flat pay W does not lead to any change in effort since it does not alter incentives to work. An increase in the cost parameter c leads to lower effort, since it makes effort (at the margin) more costly.

3. The firm maximizes
$$\max_{W,w} (p - w) e^* - W = (p - w) \frac{w}{c} - W.$$
First, it is clear that the firm wants to set W as low as possible, since W does not help to motivate workers, hence $W = 0$ (remember I wrote $W \geq 0$). The solution for w we find with the first order conditions:
$$\begin{aligned} f.o.c. &: \frac{p}{c} - \frac{2w}{c} = 0 \text{ and hence } w^* = p/2 \\ s.o.c &: -2/c < 0. \end{aligned}$$
Hence, the solution is no flat pay ($W^* = 0$) and a piece rate which compensates the workers for half of the value of the effort: $w^* = p/2$.

4. The focs are
$$\begin{aligned} w - ce + \alpha [p - w] &= 0 \\ e^*_\alpha &= \frac{w + \alpha [p - w]}{c}. \end{aligned}$$
The second order conditions are fine again:
$$-c < 0$$

5. Notice that since $p > w$ (one can derive this in principle) and $\alpha > 0$, it follows that e^*_α is larger than e^*. This is not surprising: the altruistic worker cares about the output for the firm, and hence is motivated to work harder, beyond what they would do purely based on the piece rate incentives.

6. We obtain $\partial e^*/\partial p = 0$ and $\partial e^*_\alpha/\partial p = \alpha/c$. That is, in the selfish case, the worker does not care about the return to the firm per se, holding constant the piece rate (the firm, of course, would set the piece rate differently as a function of p, as we saw in point 3, but that is a different story). The altruistic workers instead responds to a higher value of effort for the firm by working harder.

7. We obtain $\partial e^*/\partial w = 1/c$ and $\partial e^*_\alpha/\partial w = (1 - \alpha)/c$. That is, the altruistic worker responds *less* to the piece rate than the non-altruistic workers. The intuition for this is as follows: the altruistic workers knows that the piece rate paid from the firm will go to lower the firm profits, and internalizes it in the effort decision. Hence, she will respond a bit less to the incentive (but work harder in general, irrespective of w)

8. The social planner program simplifies to

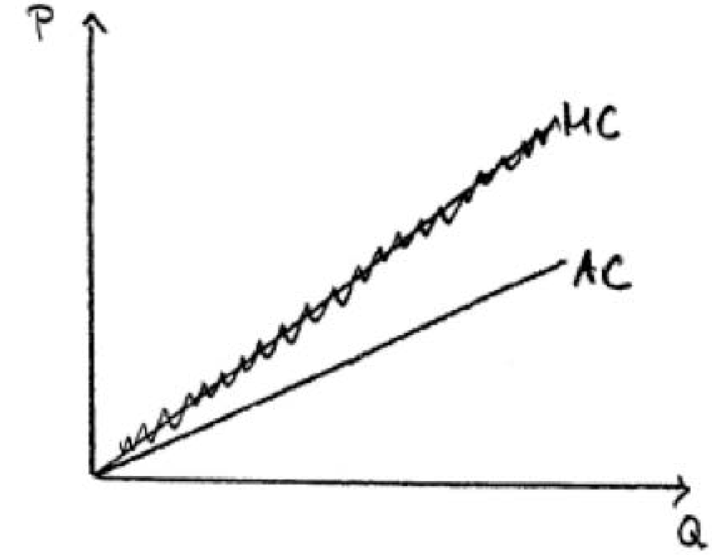
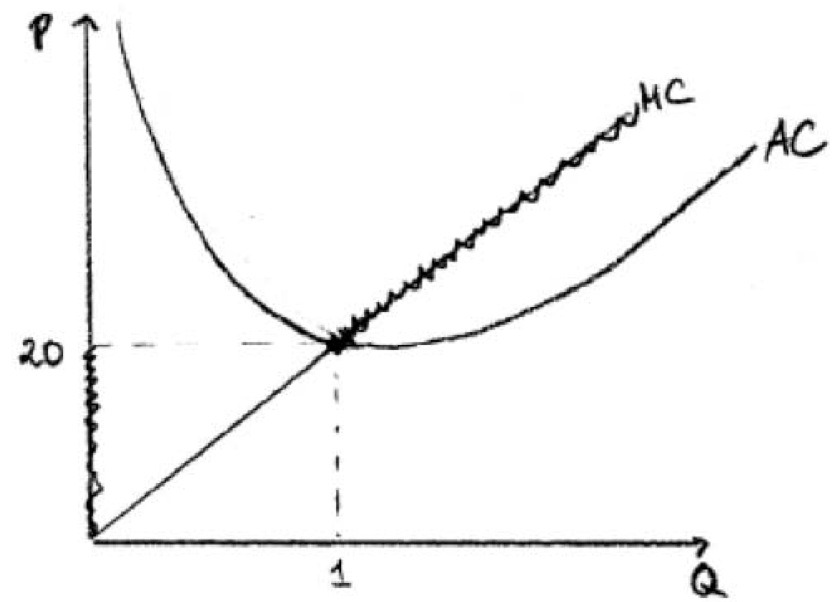
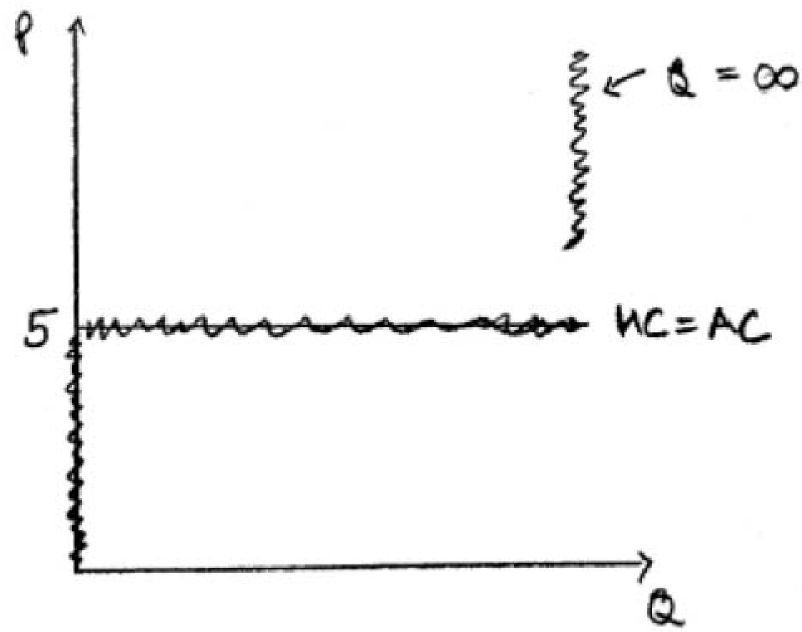
$$pe - ce^2/2$$

and hence first order conditions

$$\begin{aligned} p - ce &= 0 \\ e^*_{SP} &= p/c. \end{aligned}$$

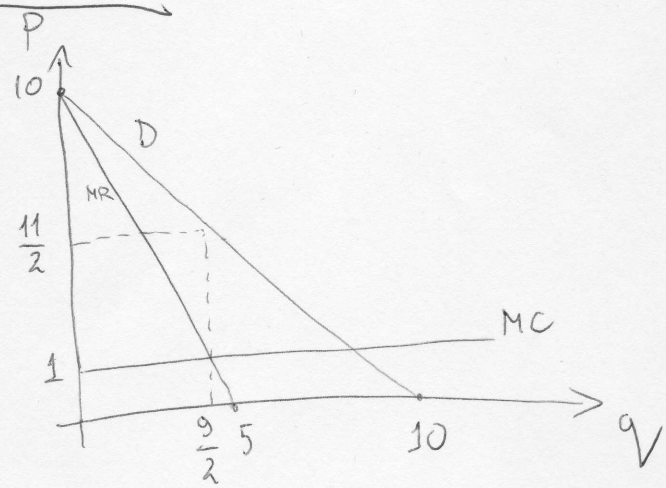
9. Comparing e^*_{SP} and e^*_{α}, it is easy to see that the two are identical for $\alpha = 1$. This is the case of perfect altruism, in which the worker cares as much about the firm as she does about herself. In this case, the worker acts as the benevolent social planner above, because she maximizes the sum of the utilities. Hence, the presence of altruism can bring the individual solution closer to the social optimum.

PROBLEM 2

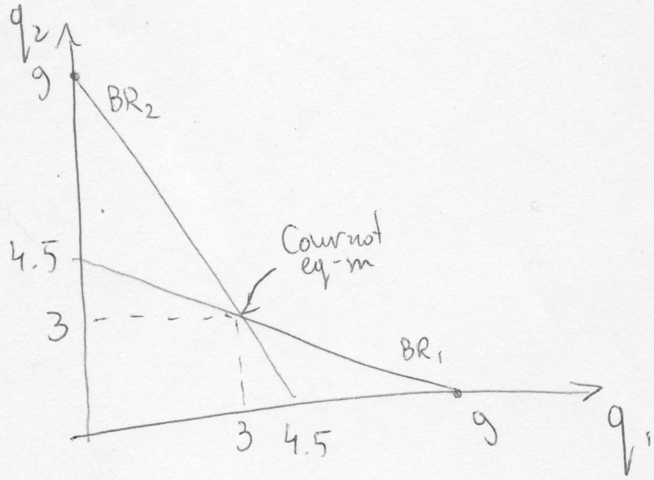
~~~ SUPPLY CURVE

① (a)



(b)



(c)


~~~

Problem 3

(2)



(5)



(8)