Econ 101A – Midterm 2
Th 2 November 2006.

You have approximately 1 hour and 20 minutes to answer the questions in the midterm. I will collect the exams at 11.00 sharp. Show your work, and good luck!

Problem 1. Profit Maximization with Uncertainty (65 points) Allison owns a company up in Napa that produces wine. Her income is given by the profits of the company that she runs. Allison is risk-averse, with utility function $u(c)$ satisfying $u'(c) > 0$ for all c, and $u''(c) < 0$ for all c. Allison maximizes her utility from the profits of the company. Wine y is a labor-intensive operation. It is produced using labor with the production function $y = l^\alpha$, with $0 < \alpha < 1$. Workers in the vine are paid the wage w. Allison maximizes utility from her only income source, the profits from wine sales:

$$\max_l u\left[pl^\alpha - wl\right]. \quad (1)$$

1. Define the concept of returns to scale, that is, what we mean by constant/increasing/decreasing returns to scale. Under what conditions for α the production of wine has decreasing returns to scale? Show the steps of your reasoning. (Here assume only $\alpha > 0$, from the next point on assume $0 < \alpha < 1$) (4 points)
2. Consider the maximization of utility of Allison, and obtain first-order conditions.(3 points)
3. Are the second order conditions satisfied? (3 points)
4. Solve for the utility-maximizing l^*. How does the optimal employment of workers in the vineyard vary as the wage w of workers increases? And when the price p increases? Discuss the economic intuition. (3 points)
5. Using the envelope theorem, compute the effect on the utility of Allison at the optimum $u\left[p\left(l^*\right)^\alpha - wl^*\right]$ of an increase in price p. Does the finding make sense? Discuss. (4 points)
6. In a nearby terrain, Wei runs his operation with a standard profit-maximization. He has the same production function and costs as Allison, and hence maximizes

   $$\max_l pl^\alpha - wl. \quad (2)$$

   Solve for the profit-maximizing l^* for Wei. (3 points)
7. Allison and Wei are neighbors to a chatty economist trained in Stanford. (a concession to Prasad) The economist, after drinking a little too much of their wine, is on-the-record on the local TV as saying "it is a pity that Allison is so risk-averse. Unlike Allison, Wei just maximizes profits, which is much better for the company". Discuss this assertion in light of what you found so far. Provide economic and mathematical intuition. (6 points)
8. The world of wine-growing is a treacherous one, with much uncertainty. In particular, the price of wine fluctuates from year-to-year, and the decisions on how much to produce and how many workers to hire have to be made before the level of price is known. The price p of a bottle is 10 with probability q and 5 with probability $1 - q$. Let's consider first the problem of Wei, (2). Wei now maximizes *expected* profits. Set up the maximization of expected profits by Wei, explaining why you are setting up the problem the way you are. (6 points)
9. Derive Wei's first order conditions and solve for the profit-maximizing level of labor hired l^* by Wei. (5 points)
10. How does the quantity of labor hired l^* vary as the probability of a high price of wine q increases? Provide economic intuition (3 points)
11. Suppose that instead of facing uncertainty over the price of wine, Wei knew that the price of wine was going to be for sure $q10 + (1 - q)5$, which is the expected value of the uncertain price. Using the solution you obtained in point 6, compute the quantity of labor hired by Wei in this case, and compare it to the quantity hired when the price is uncertain. How do the two quantities compare? (I mean, the one in point 9 and the one you just derived) Discuss the reason for this result. (5 points)

12. Let us now go back to Allison's problem (1). Allison now faces uncertainty over the price of wine as well. As in Wei's case, the price p of a bottle is 10 with probability q and 5 with probability $1-q$. Allison now maximizes *expected* utility. Set up the maximization of expected utility by Allison. Derive the first-order conditions. (6 points)

13. Solving this problem formally is tricky. Using your economic intuition, discuss whether you expect that the quantity of labor hired by Allison l^* in this case with uncertainty will be the same, smaller, or larger than the quantity hired by Wei (that is, what you derived in point 10. Compared to the discussion in point 7, can the utility function u matter here? (6 points)

14. (Hard) Can you prove formally whether the quantity of labor hired by Allison l^* in this case with uncertainty will be the same, smaller, or larger than the quantity hired by Wei? (Hint: use the property that $u'(y) < u'(x)$ for $y > x$ if u is concave) (8 points)

Solution to Problem 1.

1. The concepts of returns to scale captures the response of production to an increase of all the inputs (here, just labor). Formally, a production function with only one input, L, displays decreasing returns to scale if
$$f(tL) > tf(L) \text{ for all } L \text{ and } t > 1,$$
constant returns to scale if
$$f(tL) = tf(L) \text{ for all } L \text{ and } t > 1,$$
increasing returns to scale if
$$f(tL) < tf(L) \text{ for all } L \text{ and } t > 1.$$
In this case, $f(tL) = (tL)^\alpha = t^\alpha L^\alpha = t^\alpha f(L)$. Since $t^\alpha < t$ iff $\alpha < 1$, the function display decreasing returns to scale iff $\alpha < 1$.

2. The maximization of profits of Allison leads to the condition
$$u'\left[p(l^*)^\alpha - wl^*\right]\left(p\alpha(l^*)^{\alpha-1} - w\right) = 0.$$

3. The second order conditions are
$$u''\left[p(l^*)^\alpha - wl^*\right]\left(p\alpha(l^*)^{\alpha-1} - w\right)^2 + u'\left[p(l^*)^\alpha - wl^*\right]\left(p\alpha(\alpha-1)(l^*)^{\alpha-2}\right)$$
which are negative since $u''(x) < 0$ for all x and $\alpha < 1$ by assumption. Hence, the s.o.c. is satisfied.

4. Since $u'(pl^\alpha - wl)$ is always positive, the f.o.c. above implies
$$p\alpha(l^*)^{\alpha-1} - w = 0$$
or
$$l^* = \left(\frac{w}{\alpha p}\right)^{-1/(1-\alpha)}$$
As wage increases, the employment of workers decreases, since labor has become more costly. As the price increases, the employment of workers increases, since it has become more profitable to produce more.

5. Using the envelope theorem, we know that
$$\frac{du\left[p(l^*)^\alpha - wl^*\right]}{dp} = \frac{\partial u\left[p(l^*)^\alpha - wl^*\right]}{\partial p} = u'\left[p(l^*)^\alpha - wl^*\right] * (l^*)^\alpha > 0.$$
An increase in price p increases the profits and hence the utility of Allison in equilibrium.

6. Wei maximizes
$$\max_l pl^\alpha - wl.$$
. The f.o.c. is
$$p\alpha (l^*)^{\alpha-1} - w = 0,$$
leading to
$$l^* = \left(\frac{w}{\alpha p}\right)^{-1/(1-\alpha)},$$
the same solution as for Allison.

7. The tipsy economist is not right in this case (though he'll be closer to right for the case below). The fact that Allison is risk-averse does not affect in any way the maximization. One way to think of it is that the function $u(x)$ introduces simply a monotonic transformation of the profit function, and monotonic transformations of a function do not affect the optimal solution. (Can you prove this?) This is very similar to what we saw for the case of utility maximization: a monotonic transformation of the same utility function represents the same preferences, and hence will lead to the same utility-maximizing solution. This will no more be true once we introduce uncertainty.

8. Now that there is uncertainty about the price of wine, Wei maximizes
$$\max_l q\left[10l^\alpha - wl\right] + (1-q)\left[5l^\alpha - wl\right].$$
Expected utility tells that that we maximize the sum of the utilities in each state of the world (high price/low price), weighted by the probabilities of each state.

9. The first-order condition is
$$q\left[10\alpha l^{\alpha-1} - w\right] + (1-q)\left[5\alpha l^{\alpha-1} - w\right] = 0 \tag{3}$$
or
$$\left[q10 + (1-q)5\right]\alpha l^{\alpha-1} - w = 0$$
or
$$l^* = \left(\frac{w}{\alpha\left(q10 + (1-q)5\right)}\right)^{-1/(1-\alpha)}$$

10. As the probability that the price of wine is high increases, the quantity of labor employed increases. The higher the expected price at which Wei can sell the wine, the more Wei boosts the production. A higher (expected) price justifies hiring more workers and sending them to grow the marginal parts (higher marginal cost) of the vineyard that Wei would not use otherwise.

11. Now that there is no more uncertainty about the price of wine, we are back to the case above, and we can use the solution
$$l^* = \left(\frac{w}{\alpha p}\right)^{-1/(1-\alpha)}$$
to derive
$$l^* = \left(\frac{w}{\alpha\left(q10 + (1-q)5\right)}\right)^{-1/(1-\alpha)},$$
which is the same solution as in point 9. Wei just maximizes expected profits, he does not care whether the price is certain or uncertain, as long as the expected value of the price is the same.

12. Allison's problem is
$$\max_l qu\left[10l^\alpha - wl\right] + (1-q)u\left[5l^\alpha - wl\right].$$
This is because Allison maximizes expected utility. The first-order condition is
$$qu'\left(10l^\alpha - wl\right)\left(10\alpha l^{\alpha-1} - w\right) + (1-q)u'\left(5l^\alpha - wl\right)\left(5\alpha l^{\alpha-1} - w\right) = 0.$$

13. Allison maximizes the utility of profits, and the utility is concave. this means that Allison cares more at the margin about low profits than she cares about high profits. When setting the optimal price, Allison does not know whether prices will be high or low. If prices are high, she can make a little extra profits by increasing the quantity produced, but this will reduce the profits obtained if the price turns out to be low. Conversely, is prices are low, she can make extra profit by reducing the quantity produced, but this will reduce the profits earned it the price is high. By the concavity of the utility function, Allison cares more about decisions when profits are low (and hence utility is low) than when profits are high (and utility is high). Hence, she cares more about the case in which prices are low, and she reduces the quantity produced, relative to what Wei produces.

14. Compute the derivative of the expected utility function with respect to the amount invested s:

$$qu'\left(10l^{\alpha}-wl\right)\left(10\alpha l^{\alpha-1}-w\right)+\left(1-q\right)u'\left(5l^{\alpha}-wl\right)\left(5\alpha l^{\alpha-1}-w\right). \tag{4}$$

This expression, when set equal to zero, generates the first order condition. The trick is to prove that this expression is negative for the value of $l_W^*=\left(\frac{w}{\alpha(q10+(1-q)5)}\right)^{-1/(1-\alpha)}$ that maximizes the expected utility of Wei. This implies that Allison, unlike Wei, can improve utility by producing less than l_W^*. Together with the concavity of the utility function of Wei, this implies that the optimum for Allison l_A^* is lower than for Wei, that is, $l_A^*<l_W^*$. To prove this, note that expression (4) has the same sign as expression (we divided by $u'\left(10l^{\alpha}-wl\right)+u'\left(5l^{\alpha}-wl\right)$)

$$Bq\left(10\alpha l^{\alpha-1}-w\right)+\left(1-B\right)\left(1-q\right)\left(5\alpha l^{\alpha-1}-w\right), \tag{5}$$

with

$$B=\frac{u'\left(10l^{\alpha}-wl\right)}{u'\left(10l^{\alpha}-wl\right)+u'\left(5l^{\alpha}-wl\right)}.$$

Expression (5) is the weighted sum of $q\left(10\alpha l^{\alpha-1}-w\right)$ and of $\left(1-q\right)\left(5\alpha l^{\alpha-1}-w\right)$. Evaluate now expression (5) at $l=l_W^*$ which, by definition, satisfies Wei's first order condition (3). If we had $B=1/2$, we would go back to Wei's first order condition (3), and the expression would be zero. By the risk-aversion property, however, $B<1/2$, since $u'\left(10l^{\alpha}-wl\right)<u'\left(5l^{\alpha}-wl\right)$. Since $10\alpha l^{\alpha-1}-w>5\alpha l^{\alpha-1}-w$, it follows that, for $l=l_{W,}^*$ expression (5) is smaller than 0 since it is a convex combination that gives less weight to the term $10l^{\alpha}-wl$ and more weight to $5\alpha l^{\alpha-1}-w$, compared to expression (3) which is zero. We are done – this was a hard proof to do.

Problem 2. Investment in Risky Asset (30 points) Consider a standard problem of investment in risky assets, similar to the one that we covered in class. The agent can invest in bonds or stocks. Bonds have a return $r = 0.$ (that is, they return back your capital at the end of the period) Stocks have a stochastic return, .10 (ten percent) with probability 1/2, and $-.05$ (minus 5 percent) with probability 1/2. The agent has income w and utility function $u(x)$. The agents wants to decide the amount s of wealth to invest in stocks; we assume $0 \leq s \leq w$. The remaining of his wealth, $w - s$, is invested in bonds. (Note: s is not the share of wealth, but the amount of wealth) The agent's utility function (defined over wealth at the end of the period) is linear, $u(x) = a + bx$, with $b > 0$.

1. At the beginning, before investing, the agent has wealth w. What is the wealth at the end, after investing, if the return of the stock is high? What is the wealth if the return of the stock is low?(4 points)
2. Write down the expected utility Eu of the agent, explaining how the expected utility is defined. (4 points)
3. Derive the first order condition. (4 points)
4. Derive the second order condition. Is it satisfied? (3 points)
5. Find the solution for s^*. Argue the steps of the solution as precisely as you can. (Hint: Consider your answer to point 4) (10 points)
6. How does the solution for s^* change if bonds have a return $r = .05$ (5 percent)? (5 points)

Solution to Problem 2.

1. The agent invests $w - s$ dollars into a bond which provides gross return $(w - s)$ and s dollars into a stock which provides return $s(1 + .10)$ with probability 1/2 and $s(1 - .05)$ with probability $1 - 1/2$. Therefore, if stocks give high returns, the agent's wealth at the end is $(w - s) + s(1.1) = w + .1s$; if stocks give low returns, the agent's wealth at the end is $(w - s) + s(.95) = w - .05s$.
2. The expected utility is the probability-weighted utility under each state. Therefore, the expected utility U is
$$U = \frac{1}{2}(a + b(w + .1s)) + \frac{1}{2}(a + b(w - .05s)).$$
3. The first order condition with respect to s is
$$\frac{1}{20}b - \frac{1}{40}b = \frac{1}{40}b = 0 \tag{6}$$
4. The second order condition is 0 (not satisfied), which means that we should look for corner solutions.
5. The derivative of the utility function with respect to the investment s is $\frac{1}{40}b$ from (6). Since $b > -0$, $\frac{1}{40}b$ is also positive, and hence the utility is always increase in the level of s. The agent wants to increase the s as much as possible, that is, up to $s^* = w$, which is the solution.
6. If $r = .05$, the utility function becomes
$$\begin{aligned} U &= \frac{1}{2}(a + b((w - s)(1.05) + 1.1s)) + \frac{1}{2}(a + b((w - s)(1.05) + .95s)) = \\ &= \frac{1}{2}(a + b(1.05w + .05s)) + \frac{1}{2}(a + b(1.05w - .10s)) = \\ &= a + b * 1.05w + b\frac{.05}{2}s - b\frac{.10}{2} = \\ & \quad a + b * 1.05w - b\frac{1}{40}s. \end{aligned}$$

Differentiating this with respect to s, we get $-b/40$, which is negative. To the contrary of what we found before, here the agent always wants to lower s, which will lead to $s^* = 0$ as the solution. Intuitively, now that the return to the riskless asset has increased, the expected return on the stock is now lower than the expected return to the bond, and a risk-neutral investor will always go for the asset with higher return.