Econ 101A – Solution to Midterm 1
Th 3 October.

Problem 1. Labor supply with social comparison. (48 points) In this exercise, we consider a labor supply model with Cobb-Douglas preferences. The non-standard feature in this problem is that the preferences for consumption depend on a reference level C. Individuals in this society are only happy if they consumer more than a reference level C. You can think of this level as consumption of the neighbour. Consider the following utility function:

$$u(c,l;C) = (c-C)^{\alpha} l^{1-\alpha}$$

with $0 < \alpha < 1$ and $C > 0$. The utility from consumption of good c depends on the average consumption in society C. The price of the consumption good is p. Leisure is l.

1. How does the utility function change as C changes? In other words, compute $\partial u(c,l;C)/\partial C$. Why is this term negative? (4 points)
2. We are looking at labor supply in one day. We assume that, if the individual does not work, s/he takes leisure. In other words, the hours worked h equal $24 - l$. The hourly wage equals w. There are no sources of income other than income from hours worked. Write down the budget constraint as a function of c and l. [Hint: amount spent on the consumption good has to be smaller or equal than income earned] (6 points)
3. Write down the maximization problem of the worker with respect to c and l. Assume that the budget constraint is satisfied with equality. Why can we assume that the budget constraint is satisfied with equality? Provide as complete an explanation as you can. (10 points)
4. Write down the Lagrangean function. (2 points)
5. Write down the first order conditions for this problem with respect to c, l, and λ. (4 points)
6. Solve explicitely for c^* and l^* as a function of p, w, C, and α. (8 points)
7. Notice that the utility function $(c^* - C)^{\alpha} (l^*)^{1-\alpha}$ is defined only for $c^* > C, l^* > 0$. Show that these conditions are satisfied if $C \leq 24w/p$. From now on we assume these conditions satisfied.(2 points)
8. Use the expression for l^* that you obtained in point 6. Differentiate it with respect to C, that is, compute $\partial l^*/\partial C$. Why is this amount negative? [Hint: The individual needs to work more if...] (4 points)
9. Now calculate $\partial u(c^*(p,w), l^*(p,w))/\partial C$. Do it two ways. First, substitute into the utility function the expressions for $c^*(p,w)$ and $l^*(p,w)$ obtained at point 6, and differentiate the resulting expression with respect to C. Second, use the envelope theorem. The two results should coincide! What happens to utility at the optimum as the reference level increases? Is this result surprising? (8 points)

Solution to Problem 1.

1. $\partial u(c,l;C)/\partial C = -\alpha(c-C)^{\alpha-1} l^{1-\alpha} < 0$. The intuition here is that having a high comparison level is bad. Higher consumption C in the comparison group is associated to a lower level of utility. Comparing to someone with nicer things makes us feel bad [this may or may not apply to you].
2. The budget constraint is $pc \leq w(24-l)$ or $pc + wl \leq 24w$.
3. The maximization problem is

$$\max_{c,l} (c-C)^{\alpha} l^{1-\alpha}$$
$$\text{s.t. } pc + wl = 24w.$$

   We can write down the budget constraint with equality because the utility function is strictly increasing both in c and l. Formally, $u'_c(c,l) > 0$ and $u'_l(c,l) > 0$ for all c and l. Given that utility is strictly increasing in both consumption and leisure, the consumer will never choose a point on the interior of the budget set, i.e., a point $(\hat{c}, \hat{l})$ such that $p\hat{c} + w\hat{l} < 24w$. The reason is that the consumer could choose a point $(\hat{c}', \hat{l}')$ with $\hat{c}' > \hat{c}$ and $\hat{l}' > \hat{l}$ that still satisfies the budget constraint, i.e., such that $p\hat{c}' + w\hat{l}' \leq 24w$. (just pick $(\hat{c}', \hat{l}')$ sufficiently close to $(\hat{c}, \hat{l})$) But, given the monotonicity of u, the bundle $(\hat{c}', \hat{l}')$ provides a higher utility than the bundle $(\hat{c}, \hat{l})$. Therefore the consumer in the optimum will never choose a bundle $(\hat{c}, \hat{l})$ such that $p\hat{c} + w\hat{l} < 24w$. We can therefore limit ourselves to the points with $pc + wl = 24w$.

4. Lagrangean is $L(c, l, \lambda) = (c - C)^{\alpha} l^{1-\alpha} - \lambda (pc + wl - 24w)$.

5. First order conditions:

$$\begin{aligned} \frac{\partial L}{\partial c} &= \alpha(c^* - C)^{\alpha-1} (l^*)^{1-\alpha} - \lambda^* p = 0 \\ \frac{\partial L}{\partial l} &= (1-\alpha)(c^* - C)^{\alpha} (l^*)^{-\alpha} - \lambda^* w = 0 \\ \frac{\partial L}{\partial \lambda} &= pc^* + wl^* - 24w = 0 \end{aligned}$$

6. Using the first two first order conditions, we find

$$\frac{\alpha l^*}{(1-\alpha)(c^* - C)} = \frac{p}{w}$$

or $l^* w = \frac{(1-\alpha)}{\alpha} (c^* - C) p$. We substitute this into the budget constraint to get $pc^* + \frac{(1-\alpha)}{\alpha} (c^* - C) p - 24w = 0$ or $pc^*/\alpha = 24w + \frac{1-\alpha}{\alpha} Cp$ or

$$c^* = (1-\alpha) C + \alpha 24 \frac{w}{p}. \tag{1}$$

The demand for the consumption good is a convex combination of the reference level C and the maximal number of units that the consumer could afford, $24w/p$. Using the budget constraint, we obtain $l^* = (24w - pc^*)/w = (24w - p(1-\alpha) C - \alpha 24w)/w = (1-\alpha) 24 - (1-\alpha) pC/w$ or

$$l^* = (1-\alpha)\left[24 - \frac{p}{w} C\right]. \tag{2}$$

7. This solution makes sense if $c^* \geq C$ and $l^* \geq 0$. Using expression (1), it is clear that $c^* \geq C$ if and only if $24w/p \geq C$. The same condition guarantees $l^* \geq 0$ (use equation (2)) In words, the agent must have at least enough income to afford C units of the consumption good.

8. From expression (1) it is clear that $\partial l^* / \partial C = -(1-\alpha) p/W < 0$ for all M. If the consumption level in the neighbourhood C is higher, the individual needs to work harder (have less leisure) in order to keep up with the Joneses.

9. **First method**: we substitue the values of $c^*(p, w)$ and $l^*(p, w)$ into the utility function $u(c, l; C)$. We get

$$\begin{aligned} u(c^*, l^*; C) &= \left[(1-\alpha) C + \alpha 24 \frac{w}{p} - C\right]^{\alpha} \left[(1-\alpha)\left[24 - \frac{p}{w} C\right]\right]^{1-\alpha} = \\ &= \left[\alpha \left(24 \frac{w}{p} - C\right)\right]^{\alpha} (1-\alpha)^{1-\alpha} \left[24 - \frac{p}{w} C\right]^{1-\alpha} = \\ &= \alpha^{\alpha} (1-\alpha)^{1-\alpha} \left(24 \frac{w}{p} - C\right)^{\alpha} \left(24 \frac{w}{p} - C\right)^{1-\alpha} \left(\frac{p}{w}\right)^{1-\alpha} = \\ &= \alpha^{\alpha} (1-\alpha)^{1-\alpha} \left(24 \frac{w}{p} - C\right) \left(\frac{p}{w}\right)^{1-\alpha} \end{aligned} \tag{3}$$

Differentiate with respect to C to get

$$\frac{\partial u(c^*, l^*; C)}{\partial C} = -\alpha^{\alpha} (1-\alpha)^{1-\alpha} \left(\frac{p}{w}\right)^{1-\alpha} < 0.$$

Therefore, as the comparison level goes up (higher r_1) utility in equilibrium goes down. **Second method**: Use the envelope theorem to compute

$$\begin{aligned}\partial v(p,w;C)/\partial C &= \partial\left[(c^*-C)^{\alpha}(l^*)^{1-\alpha}-\lambda^*(pc^*+wl^*-24w)\right]/\partial C = \\ &= -\alpha(c^*-C)^{\alpha-1}(l^*)^{1-\alpha}<0\end{aligned} \tag{4}$$

since $c^* > C$ and $l^* > 0$. As you see, it is a lot faster to use the envelope theorem if all we want to know is the sign. In order to show that expressions (4) and (3) coincide, substitute in $c^*(p,w)$ and $l^*(p,w)$ into (4) to get

$$\begin{aligned}\partial v(p,w;C)/\partial C &= -\alpha\left[(1-\alpha)C+\alpha 24\frac{w}{p}-C\right]^{\alpha-1}\left[(1-\alpha)\left[24-\frac{p}{w}C\right]\right]^{1-\alpha} = \\ &= -\alpha\left[\alpha\left(24\frac{w}{p}-C\right)\right]^{\alpha-1}\left[(1-\alpha)\left[24-\frac{p}{w}C\right]\right]^{1-\alpha} = \\ &= -\alpha^{\alpha}(1-\alpha)^{1-\alpha}\left(24\frac{w}{p}-C\right)^{\alpha-1}\left[24-\frac{p}{w}C\right]^{1-\alpha} = \\ &= -\alpha^{\alpha}(1-\alpha)^{1-\alpha}\left(\frac{p}{w}\right)^{1-\alpha}.\end{aligned}$$

That was a lot of calculations – sorry! You were not expected to do all of that. What is an important point, and you learnt it the hard way with this exam, is that it is much easier to compute the sign of $\partial v(p,w;C)/\partial C$ using the envelope theorem (the second way) than doing the substitution (the first way). After all, it was not so hard to get expression (4). And most of the time this is all we care about, the sign of the effect.

Problem 2. Short answers. (12 points) In this part, you are required to provide short answers to the following two questions:

1. Consider the set $\{a,b,c,d\}$ with the following preferences defined on it: $a \succeq b$, $b \succeq c$, $c \succeq d$, $d \succ a$. Are these preferences transitive? (6 points)
2. Consider the implicit function $y - \exp(x*y) = 0$. Use the implicit function theorem to write down $\partial y/\partial x$. (assume that all the assumption needed to apply the theorem are satisfied) (6 points)

Solution to Problem 2.

1. Suppose that the preferences above are transitive. Then $a \succeq b$ and $b \succeq c$ imply $a \succeq c$. Use transitivity again on $a \succeq c$ and $c \succeq d$ to conclude $a \succeq d$. But this contradicts $d \succ a$, which by definition means NOT($a \succeq d$). Contradiction. Therefore, the preferences are not transitive.
2. Use the implicit function theorem:

$$\partial y/\partial x = -\frac{-\exp(x*y)y}{1-\exp(x*y)x} = \frac{\exp(x*y)y}{1-\exp(x*y)x}$$