Outline

1. Common utility functions

2. Utility maximization

3. Utility maximization – Tricky Cases

4. Indirect Utility Function
1 Common utility functions

- Nicholson, Ch. 3, pp. 102-105

1. Cobb-Douglas preferences: \( u(x_1, x_2) = x_1^{\alpha}x_2^{1-\alpha} \)
   - \( MRS = -\alpha x_1^{\alpha-1}x_2^{1-\alpha}/(1-\alpha)x_1^{\alpha}x_2^{-\alpha} = -\frac{\alpha}{1-\alpha} \frac{x_2}{x_1} \)

2. Perfect substitutes: \( u(x_1, x_2) = \alpha x_1 + \beta x_2 \)
   - \( MRS = -\alpha/\beta \)
3. Perfect complements: \( u(x_1, x_2) = \min(\alpha x_1, \beta x_2) \)

- \( MRS \) discontinuous at \( x_2 = \frac{\alpha}{\beta} x_1 \)

4. Constant Elasticity of Substitution: \( u(x_1, x_2) = \left( \alpha x_1^\rho + \beta x_2^\rho \right)^{1/\rho} \)

- \( MRS = -\frac{\alpha}{\beta} \left( \frac{x_1}{x_2} \right)^{\rho-1} \)

- if \( \rho = 1 \), then...

- if \( \rho = 0 \), then...

- if \( \rho \to -\infty \), then...
2 Utility Maximization

- Nicholson, Ch. 4, pp. 119–128

- \( X = R^2_+ \) (2 goods)

- Consumers: choose bundle \( x = (x_1, x_2) \) in \( X \) which yields highest utility.

- Constraint: income = \( M \)

- Price of good 1 = \( p_1 \), price of good 2 = \( p_2 \)

- Bundle \( x \) is feasible if \( p_1 x_1 + p_2 x_2 \leq M \)

- Consumer maximizes

\[
\max_{x_1, x_2} u(x_1, x_2) \\
\text{s.t. } p_1 x_1 + p_2 x_2 \leq M \\
x_1 \geq 0, \; x_2 \geq 0
\]
Maximization subject to inequality. How do we solve that?

Trick: \( u \) strictly increasing in at least one dimension. (\( \preceq \) strictly monotonic)

Budget constraint always satisfied with equality

Ignore temporarily \( x_1 \geq 0, x_2 \geq 0 \) and check afterwards that they are satisfied for \( x_1^* \) and \( x_2^* \).
• Problem becomes

\[
\max_{x_1, x_2} u(x_1, x_2)
\]
\[
s.t. \quad p_1 x_1 + p_2 x_2 - M = 0
\]

• \(L(x_1, x_2) = u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - M)\)

• F.o.c.s:

\[
\frac{u'}{x_i} - \lambda p_i = 0 \quad \text{for } i = 1, 2
\]
\[
p_1 x_1 + p_2 x_2 - M = 0
\]
• Moving the two terms across and dividing, we get:

\[
MRS = -\frac{u'_{x_1}}{u'_{x_2}} = -\frac{p_1}{p_2}
\]

• Graphical interpretation.
• Second order conditions:

\[ H = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u''_{x_1,x_1} & u''_{x_1,x_2} \\ -p_2 & u''_{x_2,x_1} & u''_{x_2,x_2} \end{pmatrix} \]

\[ |H| = p_1 (-p_1 u''_{x_2,x_2} + p_2 u''_{x_2,x_1}) - p_2 (-p_1 u''_{x_1,x_2} + p_2 u''_{x_1,x_1}) \]
\[ = -p_1^2 u''_{x_2,x_2} + 2p_1 p_2 u''_{x_1,x_2} - p_2^2 u''_{x_1,x_1} \]

• Notice: \( u''_{x_2,x_2} < 0 \) and \( u''_{x_1,x_1} < 0 \) usually satisfied (but check it!).

• Condition \( u''_{x_1,x_2} > 0 \) is then sufficient
• Example with CES utility function.
\[
\max_{x_1, x_2} \left( \alpha x_1^\rho + \beta x_2^\rho \right)^{1/\rho} \\
\text{s.t. } p_1 x_1 + p_2 x_2 - M = 0
\]

• Lagrangean =

• F.o.c.:

• Solution:
\[
x_1^* = \frac{M}{p_1 \left( 1 + \left( \frac{\alpha}{\beta} \right)^{1/(\rho-1)} \left( \frac{p_2}{p_1} \right)^{\rho/(\rho-1)} \right)}
\]
\[
x_2^* = \frac{M}{p_2 \left( 1 + \left( \frac{\beta}{\alpha} \right)^{1/(\rho-1)} \left( \frac{p_1}{p_2} \right)^{\rho/(\rho-1)} \right)}
\]
• Special case 1: $\rho = 0$ (Cobb-Douglas)

$$x_1^* = \frac{\alpha M}{\alpha + \beta p_1}$$

$$x_2^* = \frac{\beta M}{\alpha + \beta p_2}$$

• Special case 1: $\rho \to 1^-$ (Perfect Substitutes)

$$x_1^* = \begin{cases} 
0 & \text{if } p_1/p_2 \geq \alpha/\beta \\
M/p_1 & \text{if } p_1/p_2 < \alpha/\beta 
\end{cases}$$

$$x_2^* = \begin{cases} 
M/p_2 & \text{if } p_1/p_2 \geq \alpha/\beta \\
0 & \text{if } p_1/p_2 < \alpha/\beta 
\end{cases}$$
• Special case 1: $\rho \rightarrow -\infty$ (Perfect Complements)

$$x_1^* = \frac{M}{p_1 + p_2} = x_2^*$$

• Parameter $\rho$ indicates substitution pattern between goods:
  
  - $\rho > 0 \implies$ Goods are (net) substitutes
  
  - $\rho < 0 \implies$ Goods are (net) complements
3 Utility maximization – tricky cases

1. Non-convex preferences. Example:
2. Example with CES utility function.

\[
\max_{x_1,x_2} \left( \alpha x_1^\rho + \beta x_2^\rho \right)^{1/\rho} \\
\text{s.t. } p_1 x_1 + p_2 x_2 - M = 0
\]

- With \( \rho > 1 \) the interior solution is a minimum!

- Draw indifference curves for \( \rho = 1 \) (boundary case) and \( \rho = 2 \)

- Can also check using second order conditions
2. Solution does not satisfy $x_1^* > 0$ or $x_2^* > 0$. Example:

$$\max x_1 * (x_2 + 5)$$

$$\text{s.t. } p_1 x_1 + p_2 x_2 = M$$

- In this case consider corner conditions: what happens for $x_1^* = 0$? And $x_2^* = 0$?
3. Multiplicity of solutions. Example:

- Convex preferences that are not strictly convex
4 Indirect utility function

- Nicholson, Ch. 4, pp. 128-130

- Define the indirect utility \( v(p, M) \equiv u(x^*(p, M)) \), with \( p \) vector of prices and \( x^* \) vector of optimal solutions.

- \( v(p, M) \) is the utility at the optimum for prices \( p \) and income \( M \)

- Some comparative statics: \( \partial v(p, M)/\partial M =? \)

- Hint: Use Envelope Theorem on Lagrangean function
• What is the sign of $\lambda$?

• $\lambda = u'_{x_i}/p > 0$

• $\partial v(p, M)/\partial p_i = ?$

• Properties:
  
  – Indirect utility is always increasing in income $M$

  – Indirect utility is always decreasing in the price $p_i$
5 Next Class

- Comparative Statics:
  - with respect to price
  - with respect to income