

# Economics 101A

## (Lecture 3)

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## Outline

1. Implicit Function Theorem
2. Envelope Theorem
3. Convexity and concavity
4. Constrained Maximization

# 1 Implicit function theorem

- Implicit function: Ch. 2, pp. 31-32
- Consider function  $x_2 = g(x_1, p)$
- Can rewrite as  $x_2 - g(x_1, p) = 0$
- **Implicit function** has form:  $h(x_2, x_1, p) = 0$
- Often we need to go from implicit to explicit function
  
- Example 3:  $1 - x_1 * x_2 - e^{x_2} = 0$ .
- Write  $x_1$  as function of  $x_2$ :
- Write  $x_2$  as function of  $x_1$ :

- **Univariate implicit function theorem (Dini):** Consider an equation  $f(p, x) = 0$ , and a point  $(p_0, x_0)$  solution of the equation. Assume:
  1.  $f$  continuously differentiable in a neighbourhood of  $(p_0, x_0)$ ;
  2.  $f'_x(p_0, x_0) \neq 0$ .
- Then:
  1. There is one and only function  $x = g(p)$  defined in a neighbourhood of  $p_0$  that satisfies  $f(p, g(p)) = 0$  and  $g(p_0) = x_0$ ;
  2. The derivative of  $g(p)$  is

$$g'(p) = -\frac{f'_p(p, g(p))}{f'_x(p, g(p))}$$

- Example 3 (continued):  $1 - x_1 * x_2 - e^{x_2} = 0$
- Find derivative of  $x_2 = g(x_1)$  implicitly defined for  $(x_1, x_2) = (1, 0)$
- Assumptions:
  1. Satisfied?
  2. Satisfied?
- Compute derivative

### **Multivariate implicit function theorem (Dini):**

Consider a set of equations  $(f_1(p_1, \dots, p_n; x_1, \dots, x_s) = 0; \dots; f_s(p_1, \dots, p_n; x_1, \dots, x_s) = 0)$ , and a point  $(p_0, x_0)$  solution of the equation. Assume:

1.  $f_1, \dots, f_s$  continuously differentiable in a neighbourhood of  $(p_0, x_0)$ ;
2. The following Jakobian matrix  $\frac{\partial \mathbf{f}}{\partial \mathbf{x}}$  evaluated at  $(p_0, x_0)$  has determinant different from 0:

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_s} \\ \dots & \dots & \dots \\ \frac{\partial f_s}{\partial x_1} & \dots & \frac{\partial f_s}{\partial x_s} \end{pmatrix}$$

• Then:

1. There is one and only set of functions  $x = \mathbf{g}(p)$  defined in a neighbourhood of  $p_0$  that satisfy  $\mathbf{f}(p, \mathbf{g}(p)) = \mathbf{0}$  and  $\mathbf{g}(p_0) = x_0$ ;
2. The partial derivative of  $x_i$  with respect to  $p_k$  is

$$\frac{\partial g_i}{\partial p_k} = - \frac{\det \left( \frac{\partial(f_1, \dots, f_s)}{\partial(x_1, \dots, x_{i-1}, p_k, x_{i+1}, \dots, x_s)} \right)}{\det \left( \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right)}$$

- Example 2 (continued): Max  $h(x_1, x_2) = p_1 * x_1^2 + p_2 * x_2^2 - 2x_1 - 5x_2$
- f.o.c.  $x_1 : 2p_1 * x_1 - 2 = 0 = f_1(p, x)$
- f.o.c.  $x_2 : 2p_2 * x_2 - 5 = 0 = f_2(p, x)$
- Comparative statics of  $x_1^*$  with respect to  $p_1$ ?
- First compute  $\det \left( \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right)$

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$



- Then compute  $\det \left( \frac{\partial(f_1, \dots, f_s)}{\partial(x_1, \dots, x_{i-1}, p_k, x_{i+1}, \dots, x_s)} \right)$

$$\begin{pmatrix} \frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial p_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix} = \begin{pmatrix} & \\ & \end{pmatrix}$$

- Finally,  $\frac{\partial x_1}{\partial p_1} =$

- Why did you compute  $\det \left( \frac{\partial \mathbf{f}}{\partial \mathbf{x}} \right)$  already?

## 2 Envelope Theorem

- Ch. 2, pp. 35-39
- You now know how  $x_1^*$  varies if  $p_1$  varies.
- How does  $h(\mathbf{x}^*(\mathbf{p}))$  vary as  $p_1$  varies?
- Differentiate  $h(x_1^*(p_1, p_2), x_2^*(p_1, p_2), p_1, p_2)$  with respect to  $p_1$ :

$$\begin{aligned} & \frac{dh(x_1^*(p_1, p_2), x_2^*(p_1, p_2), p_1, p_2)}{dp_1} \\ &= \frac{\partial h(\mathbf{x}^*, \mathbf{p})}{\partial x_1} * \frac{\partial x_1^*(\mathbf{p})}{\partial p_1} \\ & \quad + \frac{\partial h(\mathbf{x}^*, \mathbf{p})}{\partial x_2} * \frac{\partial x_2^*(\mathbf{p})}{\partial p_1} \\ & \quad + \frac{\partial h(\mathbf{x}^*, \mathbf{p})}{\partial p_1} \end{aligned}$$

- The first two terms are zero.

- **Envelope Theorem** for unconstrained maximization. Assume that you maximize function  $f(\mathbf{x}; \mathbf{p})$  with respect to  $x$ . Consider then the function  $f$  at the optimum, that is,  $f(\mathbf{x}^*(\mathbf{p}), \mathbf{p})$ . The total differential of this function with respect to  $p_i$  equals the partial derivative with respect to  $p_i$ :

$$\frac{df(\mathbf{x}^*(\mathbf{p}), \mathbf{p})}{dp_i} = \frac{\partial f(\mathbf{x}^*(\mathbf{p}), \mathbf{p})}{\partial p_i}.$$

- You can disregard the indirect effects. Graphical intuition.

### 3 Convexity and concavity

- Function  $f$  from  $C \subset R^n$  to  $R$  is concave if

$$f(tx + (1 - t)y) \geq tf(x) + (1 - t)f(y)$$

for all  $x, y \in C$  and for all  $t \in [0, 1]$

- Notice:  $C$  must be convex set, i.e., if  $x \in C$  and  $y \in C$ , then  $tx + (1 - t)y \in C$ , for  $t \in [0, 1]$

- Function  $f$  from  $C \subset R^n$  to  $R$  is strictly concave if

$$f(tx + (1 - t)y) > tf(x) + (1 - t)f(y)$$

for all  $x, y \in C$  and for all  $t \in (0, 1)$

- Function  $f$  from  $R^n$  to  $R$  is convex if  $-f$  is concave.

- Alternative characterization of convexity.
- A function  $f$ , twice differentiable, is concave if and only if **for all**  $x$  the subdeterminants  $|H_i|$  of the Hessian matrix have the property  $|H_1| \leq 0$ ,  $|H_2| \geq 0$ ,  $|H_3| \leq 0$ , and so on.
- For the univariate case, this reduces to  $f'' \leq 0$  for all  $x$
- For the bivariate case, this reduces to  $f''_{x,x} \leq 0$  and  $f''_{x,x} * f''_{y,y} - (f''_{x,y})^2 \geq 0$
- A twice-differentiable function is strictly concave if the same property holds with strict inequalities.

- Examples.

1. For which values of  $a, b,$  and  $c$  is  $f(x) = ax^3 + bx^2 + cx + d$  is the function concave over  $R$ ?  
Strictly concave? Convex?

2. Is  $f(x, y) = -x^2 - y^2$  concave?

- For Example 2, compute the Hessian matrix

–  $f'_x =$  ,  $f'_y =$

–  $f''_{x,x} =$  ,  $f''_{x,y} =$

–  $f''_{y,x} =$  ,  $f''_{y,y} =$

– Hessian matrix  $H$  :

$$H = \begin{pmatrix} f''_{x,x} = & f''_{x,y} = \\ f''_{y,x} = & f''_{y,y} = \end{pmatrix}$$

- Compute  $|H_1| = f''_{x,x}$  and  $|H_2| = f''_{x,x} * f''_{y,y} - (f''_{x,y})^2$

- Why are convexity and concavity important?
- **Theorem.** Consider a twice-differentiable concave (convex) function over  $C \subset \mathbb{R}^n$ . If the point  $\mathbf{x}_0$  satisfies the first order conditions, it is a global maximum (minimum).
- For the proof, we need to check that the second-order conditions are satisfied.
- These conditions are satisfied by definition of concavity!
- (We have only proved that it is a local maximum)

## 4 Constrained maximization

- Ch. 2, pp. 39-45
- So far unconstrained maximization on  $\mathbb{R}$  (or open subsets)
- What if there are constraints to be satisfied?
- Example 1:  $\max_{x,y} x * y$  subject to  $3x + y = 5$
- Substitute it in:  $\max_{x,y} x * (5 - 3x)$
- Solution:  $x^* =$
- Example 2:  $\max_{x,y} xy$  subject to  $x \exp(y) + y \exp(x) = 5$
- Solution: ?



- Graphical intuition on general solution.
- Example 3:  $\max_{x,y} f(x, y) = x * y$  s.t.  $h(x, y) = x^2 + y^2 - 1 = 0$
- Draw  $0 = h(x, y) = x^2 + y^2 - 1$ .
- Draw  $x * y = K$  with  $K > 0$ . Vary  $K$
- Where is optimum?
- Where  $dy/dx$  along curve  $xy = K$  equals  $dy/dx$  along curve  $x^2 + y^2 - 1 = 0$
- Write down these slopes.

- Idea: Use implicit function theorem.
- Heuristic solution of system

$$\begin{aligned} & \max_{x,y} f(x, y) \\ & \text{s.t. } h(x, y) = 0 \end{aligned}$$

- Assume:
  - continuity and differentiability of  $h$
  - $h'_y \neq 0$  (or  $h'_x \neq 0$ )
- Implicit function Theorem: Express  $y$  as a function of  $x$  (or  $x$  as function of  $y$ )!

- Write system as  $\max_x f(x, g(x))$
- f.o.c.:  $f'_x(x, g(x)) + f'_y(x, g(x)) * \frac{\partial g(x)}{\partial x} = 0$
- What is  $\frac{\partial g(x)}{\partial x}$ ?
- Substitute in and get:  $f'_x(x, g(x)) + f'_y(x, g(x)) * (-h'_x/h'_y) = 0$  or

$$\frac{f'_x(x, g(x))}{f'_y(x, g(x))} = \frac{h'_x(x, g(x))}{h'_y(x, g(x))}$$

- **Lagrange Multiplier Theorem, necessary condition.** Consider a problem of the type

$$\begin{array}{l} \max_{x_1, \dots, x_n} f(x_1, x_2, \dots, x_n; \mathbf{p}) \\ \text{s.t.} \quad \left\{ \begin{array}{l} h_1(x_1, x_2, \dots, x_n; \mathbf{p}) = 0 \\ h_2(x_1, x_2, \dots, x_n; \mathbf{p}) = 0 \\ \dots \\ h_m(x_1, x_2, \dots, x_n; \mathbf{p}) = 0 \end{array} \right. \end{array}$$

with  $n > m$ . Let  $\mathbf{x}^* = \mathbf{x}^*(\mathbf{p})$  be a local solution to this problem.

- Assume:
  - $f$  and  $h$  differentiable at  $x^*$
  - the following Jacobian matrix at  $\mathbf{x}^*$  has maximal rank

$$J = \begin{pmatrix} \frac{\partial h_1}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial h_1}{\partial x_n}(\mathbf{x}^*) \\ \dots & \dots & \dots \\ \frac{\partial h_m}{\partial x_1}(\mathbf{x}^*) & \dots & \frac{\partial h_m}{\partial x_n}(\mathbf{x}^*) \end{pmatrix}$$

- Then, there exists a vector  $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_m)$  such that  $(\mathbf{x}^*, \boldsymbol{\lambda})$  maximize the Lagrangean function

$$L(\mathbf{x}, \boldsymbol{\lambda}) = f(\mathbf{x}; \mathbf{p}) - \sum_{j=0}^m \lambda_j h_j(\mathbf{x}; \mathbf{p})$$

- Case  $n = 2, m = 1$ .
- First order conditions are

$$\frac{\partial f(\mathbf{x}; \mathbf{p})}{\partial x_i} - \lambda \frac{\partial h(\mathbf{x}; \mathbf{p})}{\partial x_i} = 0$$

for  $i = 1, 2$

- Rewrite as

$$\frac{f'_{x_1}}{f'_{x_2}} = \frac{h'_{x_1}}{h'_{x_2}}$$

## Constrained Maximization, Sufficient condition for the case $n = 2, m = 1$ .

- If  $\mathbf{x}^*$  satisfies the Lagrangean condition, and the determinant of the bordered Hessian

$$H = \begin{pmatrix} 0 & -\frac{\partial h}{\partial x_1}(\mathbf{x}^*) & -\frac{\partial h}{\partial x_2}(\mathbf{x}^*) \\ -\frac{\partial h}{\partial x_1}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_1^2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_2 \partial x_1}(\mathbf{x}^*) \\ -\frac{\partial h}{\partial x_2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_1 \partial x_2}(\mathbf{x}^*) & \frac{\partial^2 L}{\partial x_2^2}(\mathbf{x}^*) \end{pmatrix}$$

is positive, then  $\mathbf{x}^*$  is a constrained maximum.

- If it is negative, then  $\mathbf{x}^*$  is a constrained minimum.
- Why? This is just the Hessian of the Lagrangean  $L$  with respect to  $\lambda$ ,  $x_1$ , and  $x_2$

- Example 4:  $\max_{x,y} x^2 - xy + y^2$  s.t.  $x^2 + y^2 - p = 0$

- $\max_{x,y,\lambda} x^2 - xy + y^2 - \lambda(x^2 + y^2 - p)$

- F.o.c. with respect to  $x$ :

- F.o.c. with respect to  $y$ :

- F.o.c. with respect to  $\lambda$ :

- Candidates to solution?

- Maxima and minima?

# 5 Next Class

- Next class:
  - More on Constrained Maximization
  - Preferences