Econ 101A – Midterm 2
Tu 10 November 2009.

You have approximately 1 hour and 20 minutes to answer the questions in the midterm. I will collect the exams at 11.00 sharp. Show your work, and good luck!

Problem 1. Cost Minimization. (40 points) Consider a company in a perfectly competitive industry with a familiar Cobb-Douglas production function, $y = f(L, K) = AL^{\alpha}K^{\beta}$. The company faces input costs w for labor and r for capital, and sells the final product for price p.

1. Write down the one-step profit maximization. Remember, the company is choosing L and K to maximize profits, that is, revenue minus input costs. (5 points)
2. Derive the first-order conditions. (5 points)
3. Solve for $L^* = L^*(p, w, r, A)$ and $K^* = K^*(p, w, r, A)$. (10 points)
4. Comment on how L^* varies as each of the four parameter (p, w, r, A) varies. If the derivative is obvious, just point to the sign. Explain the intuition. (10 points)
5. Without solving for the second-order conditions, explain intuitively why the second order conditions are satisfied only if $\alpha + \beta < 1$. (10 points)

Solution to Problem 1. Cost Minimization.

6.

$$\max_{L,K} py - c(y)$$
$$\max_{L,K} p[AL^{\alpha}K^{\beta}] - wL - rK$$

7.

$$\begin{aligned} focL &: \ p\alpha L^{\alpha-1}K^{\beta} \ - w = 0 \\ focK &: \ p\beta L^{\alpha}K^{\beta-1} \ - r = 0 \end{aligned}$$

8. By dividing $focL$ by $focK$ we obtain

$$\begin{aligned} \frac{\alpha K}{\beta L} &= \frac{w}{r} \\ K &= \frac{w\beta}{r\alpha}L \end{aligned}$$

We plug this back into either of the focs to get

$$\begin{aligned} L^*(w,r,p,A) &= \left[\frac{1}{pA}\left(\frac{r}{\beta}\right)^{\beta}\left(\frac{w}{\alpha}\right)^{1-\beta}\right]^{\frac{1}{\alpha+\beta-1}} \\ K^*(w,r,p,A) &= \left[\frac{1}{pA}\left(\frac{r}{\beta}\right)^{1-\alpha}\left(\frac{w}{\alpha}\right)^{\alpha}\right]^{\frac{1}{\alpha+\beta-1}} \end{aligned}$$

9. Using the expression for L^* found above, and the assumption that $\alpha + \beta < 1$:

$$\begin{aligned} \frac{dL^*}{dp} \propto \frac{-1}{\alpha+\beta-1} &> 0 \\ \frac{dL^*}{dw} \propto \frac{1-\beta}{\alpha+\beta-1} &< 0 \\ \frac{dL^*}{dr} \propto \frac{\beta}{\alpha+\beta-1} &< 0 \\ \frac{dL^*}{dA} \propto \frac{-1}{\alpha+\beta-1} &> 0 \end{aligned}$$

Note that these comparative statics are very different from what we would obtain if we were using the contingent factor demand functions $L^*(w,r,A,y)$ and $K^*(w,r,A,y)$ (These are the solutions to the cost minimization problem, contingent on producing exact quantity y).

L^* is increasing in p because it is optimal to produce until price equals marginal cost. If the price of its output increases, the firm will want to increase production, which means using more inputs.

L^* is decreasing in w due to two reasons: first, as the cost of labor increases, the firm will be able to produce more efficiently by using more capital (substitution effect). Second, as its total costs increase, the firm must reduce production so that its marginal cost is once again equal to p.

L^* is decreasing in r, which means that any subsitution towards labor (which is now relatively cheaper, compared to capital) is overpowered by the effects of a decrease in production: decreased use of both inputs. This means L and K are *gross complements* in production.

L^* is increasing in A because A increases the marginal productivity of both inputs, and decreases marginal cost. As a result, the firm will choose to expand production, using more inputs.

10. The second order conditions for the one-step profit maximization are (1) $f''_{LL} < 0$ and (2) $f''_{LL} f''_{KK} - f''_{LK} > 0$ (from the Hessian matrix conditions). For the Cobb-Douglas production function, these inequalities are equivalent to $\alpha + \beta < 1$. However, an intuitive argument like the one below is sufficient for this problem.

    $\alpha + \beta < 1$ corresponds to the case of decreasing returns to scale, in which a firm has increasing marginal costs and will produce only until its marginal costs are equal to the price of its output. In this case, the firm has a finite optimal production level at any price, and hence, finite factor demands $L^*$and K^*. In contrast, $\alpha + \beta < 1$ corresponds to increasing returns to scale, in which there is no finite level of production that maximizes profits (the firm would always prefer to produce more up to infinity). Similarly, when $\alpha + \beta = 1$ and we have constant returns to scale, the optimal level of production is either zero (if $p < MC$), infinity (if $p > MC$), or any quantity from zero to infinity (when $p = MC$), so we won't have finite positive demand functions at any price. Thus, factor demand functions can only exist when $\alpha + \beta < 1$, so this condition must be required for the second order conditions to hold.

Problem 2. Extended Warranties (90 points + 10 extra) When purchasing electronics at the big-box store Okay Buy, consumers are given the opportunity to purchase a one-year warranty after they purchase any product. A one-year warranty is a full insurance policy against malfunction of the product: If it stops working within one year, the consumer gets a free replacement only if they purchased the warranty. The product is an MP3 player which sells at price \$100 and that has a probability of malfunction within one year of 1/10. Consumers have utility function $u(w_0) + v$, where $w_0 = w - 100$ is the wealth *after* purchasing the cell phone, at the time when she is faced with the decision of whether to buy a warranty, and $v > 0$ is the utility of owning a functioning player. Assume $u' > 0$ and $u'' < 0$.

Consumer really need to use this product for one whole year (that is, v is extremely large). If the MP3 player malfunctions and the consumer does not have a warranty, she will buy immediately a replacement which also costs \$100. We will assume that this replacement player will not fail.

1. If the consumer *does not* purchase the warranty, what is her expected utility? What if she *does* purchase the warranty (at price p)? (4 points)

2. Derive an expression for $\overline{p(w_0)}$, the maximum price that a consumer with wealth w_0 will be willing to pay for the warranty (Do not try to solve explicitly for $\overline{p(w_0)}$). (Hint: This price can be obtained by equating the utilities in the case of purchase and the case of no purchase. Why?) (5 points)

3. From this point on assume that the utility function $u(c)$ is the exponential utility function $-\exp(-\rho c)$. Do a qualitative plot and show that ρ is the absolute risk aversion coefficient. (6 points)

4. Is it a problem that the utility $u(c) = -\exp(-\rho c)$ is always negative? Explain (5 points)

5. Show that $\overline{p(w_0)}$ implicitly defined in point 2 (and now assuming $u(c) = -\exp(-\rho c)$) does not depend on w_0 (If you get stuck here, move on to the next part of the problem). (10 points)

6. Given that $\bar{p}$ (the highest price that a consumer is willing to pay for a warranty) does not depend on w_0, derive for a representative consumer the demand function, that is, the demand for a warranty as a function of the price of the warranty p, that is, $D(p)$ (Remember that the demand D in this case can only be 0 or 1.) (5 points)

7. Assume now that there are N consumers of varying wealth levels who have purchased the MP3 player. Derive the aggregate demand curve for warranties $AD(p)$ analytically (that is, show the expression) and plot this on a graph with p on the vertical axis and quantity on the horizontal axis. (10 points)

8. Now we will consider how Okay Buy chooses prices of its one-year warranty of MP3 players. Okay Buy acts as a monopolist in providing this warranty (If you buy your MP3 player there, you can't get your warranty elsewhere.). First, determine Okay Buy's expected cost function $E[C(y)]$ for providing y extended warranties. Assume that it costs c (with $c \leq 100$) to the company to replace one unit that fails. (5 points)

9. Write the profit maximization problem that will determine how Okay Buy sets p, the price of the warranty (Remember that $AD(p)$ is the aggregate demand function). Solve for the profit-maximizing

price p_M^* and quantity y_M^*. Do not use Lagrangeans, reason through the different cases for p. (Note: You can assume that consumers who are indifferent between buying or not buying the warranty will buy it.) (10 points)

10. Why do we know that profits will be positive? (5 points)

11. Relate this answer to whether Okay Buy offers a "fair" price for this warranty. Explain what the fair price is in this case (5 points)

12. (Extra credit, do the next two questions before) Now, derive the solution for the monopolist graphically. Plot the Demand curve $AD(p)$ (you already did), the marginal revenue curve, and the marginal expected cost curve, and find p_M^* and y_M^* (10 points)

13. Now, the government comes in and introduces competition in the market for extended warranties. Okay Buy, like every other retailer, is forced to offer extended warranties by competitors. This introduces perfect competition in this market. In the presence of perfect competition, as we know, price is driven down to marginal cost. Plot the aggregate marginal cost, which in this case is the derivative of the expected cost function $E[C(y)]$ derived above. Using the graph, derive the perfect competition price p_{PC}^* and quantity sold y_{PC}^* in the whole market. (10 points)

14. Compute the consumer surplus and producer surplus in the case of monopoly and in the case of perfect competition, and compare them. (10 points)

Solution of Problem 2.

1. If she does not purchase the warranty, with probability 9/10 she has utility $u(w_0) + v$, while with probability 1/10 she has utility $u(w_0 - 100) + v$, since she will have to purchase the product again. The expected utility hence is

$$\frac{9}{10}\left[u(w_0) + v\right] + \frac{1}{10}\left[u(w_0 - 100) + v\right] = \frac{9}{10}u(w_0) + \frac{1}{10}u(w_0 - 100) + v$$

   If the agent purchases the insurance, then she will have utility $u(w_0 - p) + v$. Her utility does not depend on whether the product malfunctions, since in either case she will pay price p for the warranty, and the product will be replaced at no additional cost if it malfunctions.

2. The expression $\overline{p(w_0)}$ is the price at which the agent is indifferent between purchasing the insurance and not purchasing it. An implicit function for $\overline{p(w_0)}$ can be found by equating the two expressions found above:

$$\begin{aligned}
\frac{9}{10}u(w_0) + \frac{1}{10}u(w_0 - 100) + v &= u(w_0 - \overline{p(w_0)}) + v \text{ or} \\
\frac{9}{10}u(w_0) + \frac{1}{10}u(w_0 - 100) - u(w_0 - \overline{p(w_0)}) &= 0
\end{aligned}$$

3. We can compute the absolute risk aversion coefficient

$$r_A = -\frac{u''}{u'} = -\frac{-\rho^2 \exp(-\rho x)}{\rho \exp(-\rho x)} = \rho.$$

4. See graph attached for the shape of this negative exponential function. No, it does not matter that $u(w_0)$ is negative for all values of w_0, since the cardinal value of utility has no meaning. Utility functions are used to represent preference orderings; all we need is that if $x \succeq y$, $U(x) \geq U(y)$.

5. If the utility function $u(c)$ is the exponential utility function $-\exp(-\rho c)$, the expression above becomes

$$-\frac{9}{10}\exp(-\rho w_0) - \frac{1}{10}\exp(-\rho(w_0 - 100)) + \exp(-\rho\left(w_0 - \overline{p(w_0)}\right)) = 0.$$

We can use the implicit function theorem to compute

$$\frac{\partial\overline{p(w_0)}}{\partial w_0} = -\frac{\frac{9}{10}\rho\exp(-\rho w_0) + \frac{1}{10}\rho\exp(-\rho(w_0 - 100)) - \rho\exp(-\rho\left(w_0 - \overline{p(w_0)}\right))}{\exp(-\rho\left(w_0 - \overline{p(w_0)}\right))}$$

Notice that if we factor out $-\rho$ from the numerator, what remains in the numerator is exactly the implicit function for $\overline{p(w_0)}$ that we started out with! Since that implicit function equals zero, this proves $\frac{\partial\overline{p(w_0)}}{\partial w_0} = 0$.

6. The demand curve is

$$D(p) = \begin{cases} 1, \text{ if } p \leq \overline{p(w_0)} \\ 0, \text{ if } p > \overline{p(w_0)} \end{cases}$$

7. Since we found $\overline{p(w_0)}$ does not depend on w_0, the varying wealth levels will *not* affect consumer demand (and we can start using the simpler notation $\overline{p}$): All consumers will have the demand function of the representative consumer above. The aggregate demand curve is:

$$AD(p) = \begin{cases} N, \text{ if } p \leq \overline{p} \\ 0, \text{ if } p > \overline{p} \end{cases}$$

See attached graph of aggregate demand.

8. The cost for a warranty is 0 (no replacement) with probability 9/10 and c with probability 1/10. Hence, the expected cost for selling y warranties is $cy/10$.

9. The firm maximizes

$$\max_p pD(p) - E[C(y)] = pD(p) - cD(p)/10$$

Since the demand is discrete, we do not use derivatives but consider instead the various cases. (i) For any price $p \leq \bar{p}$ (which we now know is independent of w_0), the demand $D(p)$ equals N, and profits are $pN - cN/10 = N(p - c/10)$. (ii) For price $p > \bar{p}$, the demand $D(p) = 0$, and hence profits are zero. The firm will never want to set $p > \bar{p}$. Among the prices with $p \leq \bar{p}$, the profits $N(p - c/10)$ are increasing in p and hence the firm will set p at the highest level, that is, $p^* = \bar{p}$.

10. To show that profits are positive, we need to show that $N(\bar{p} - c/10) > 0$, or $\bar{p} > c/10$. Recall that the consumer is risk-averse (utility function is concave, with positive risk-aversion coefficient $r_A = \rho$). Therefore, by Jensen's inequality, we know $\bar{p}$ is *strictly* greater than her expected loss, which is $\frac{1}{10} \cdot \$100 = \$10$ (i.e. she is willing to pay a *risk premium* to avoid the risk of a large loss). Since we know that $c \leq \$100$, then $\bar{p} > \$10 \geq c/10$ proves that profits will be positive.

11. A fair price is the price at which $p = q$, or the amount charged for insurance is equal to the expected loss taken on by the insurer. In this particular problem, this can be interpreted two ways. From the consumer's point of view, a fair price would be $\$100/10 = \$10$, since she has a 10% chance of having to pay \$10 to replace her MP3 player. For Okay Buy, however, the fair price would be the price at which its expected profits are zero, which is the price $c/10$.

12. See attached.

13. We found in part 8 that the expected cost function is $cy/10$. This makes the marginal cost function $c/10$, and the average cost function is the same. In perfect competition, price is driven down to the minimum average cost level, which means $p^*_{PC} = c/10$. Since we found in part 10 that $\bar{p} > c/10$, we know that aggregate demand will be $Y^*_{PC} = N$ at p^*_{PC}.

14. Producer surplus is profit: $PS_M = N(\overline{p} - c/10)$ and $PS_{PC} = 0$ since price equals average cost in perfect competition.

    Consumer surplus is the area between the demand curve and the price, which is rectangular in this case: $CS_M = 0$ and $CS_{PC} = N(\overline{p} - c/10)$. Because aggregate demand is perfectly inelastic up to price $\overline{p}$, there is no deadweight loss in this problem. All the surplus goes to the producer in the monopoly scenario, and to the consumers in the perfect competition scenario.