Economics 101A
(Lecture 6)

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Outline

1. Common utility functions
2. Utility maximization
3. Utility maximization – Tricky Cases
4. Indirect Utility Function

1 Common utility functions

- Nicholson, Ch. 3, pp. 102-105

1. Cobb-Douglas preferences: $u(x_1, x_2) = x_1^{\alpha} x_2^{1-\alpha}$

   - $MRS = -\alpha x_1^{a-1} x_2^{1-\alpha} / (1-a) x_1^{\alpha} x_2^{-\alpha} = \frac{\alpha}{1-\alpha} \frac{x_2}{x_1}$

2. Perfect substitutes: $u\left(x_1, x_2\right) = \alpha x_1 + \beta x_2$

   - $MRS = -\alpha/\beta$

3. Perfect complements: $u(x_1, x_2) = \mathsf{min}(\alpha x_1, \beta x_2)$

   - MRS discontinuous at $x_2 = \frac{\alpha}{\beta} x_1$

4. Constant Elasticity of Substitution: $u(x_1, x_2) = \left(\alpha x_1^\rho + \beta x_2^\rho\right)^{1/\rho}$

   - $MRS = -\frac{\alpha}{\beta} \left(\frac{x_1}{x_2}\right)^{\rho - 1}$
   - if $\rho = 1$, then...
   - if $\rho = 0$, then...
   - if $\rho \to -\infty$, then...

2 Utility Maximization

- Nicholson, Ch. 4, pp. 119–128
- $X = R^2_+$ (2 goods)
- Consumers: choose bundle $x = (x_1, x_2)$ in X which yields highest utility.
- Constraint: income $= M$
- Price of good 1 $= p_1$, price of good 2 $= p_2$
- Bundle x is feasible if $p_1 x_1 + p_2 x_2 \leq M$
- Consumer maximizes

$$\max_{x_1, x_2} u(x_1, x_2)$$
$$s.t.\ p_1 x_1 + p_2 x_2 \leq M$$
$$x_1 \geq 0,\ x_2 \geq 0$$

- Maximization subject to inequality. How do we solve that?

- Trick: u strictly increasing in at least one dimension. ($\succeq$ strictly monotonic)

- Budget constraint always satisfied with equality

- Ignore temporarily $x_1 \geq 0$, $x_2 \geq 0$ and check afterwards that they are satisfied for x_1^* and x_2^*.

- Problem becomes

$$\max_{x_1, x_2} u(x_1, x_2)$$
$$s.t.\ p_1 x_1 + p_2 x_2 - M = 0$$

- $L(x_1, x_2) = u(x_1, x_2) - \lambda(p_1 x_1 + p_2 x_2 - M)$

- F.o.c.s:

$$\begin{aligned} u'_{x_i} - \lambda p_i &= 0 \text{ for } i = 1, 2 \\ p_1 x_1 + p_2 x_2 - M &= 0 \end{aligned}$$

- Moving the two terms across and dividing, we get:

$$MRS = -\frac{u'_{x_1}}{u'_{x_2}} = -\frac{p_1}{p_2}$$

- Graphical interpretation.

- Second order conditions:

$$H = \begin{pmatrix} 0 & -p_1 & -p_2 \\ -p_1 & u''_{x_1,x_1} & u''_{x_1,x_2} \\ -p_2 & u''_{x_2,x_1} & u''_{x_2,x_2} \end{pmatrix}$$

$$\begin{aligned} |H| &= p_1 \left(-p_1 u''_{x_2,x_2} + p_2 u''_{x_2,x_1} \right) \\ &\quad -p_2 \left(-p_1 u''_{x_1,x_2} + p_2 u''_{x_1,x_1} \right) \\ &= -p_1^2 u''_{x_2,x_2} + 2 p_1 p_2 u''_{x_1,x_2} - p_2^2 u''_{x_1,x_1} \end{aligned}$$

- Notice: $u''_{x_2,x_2} < 0$ and $u''_{x_1,x_1} < 0$ usually satisfied (but check it!).

- Condition $u''_{x_1,x_2} > 0$ is then sufficient

- Example with CES utility function.

$$\max_{x_1,x_2} \left(\alpha x_1^\rho + \beta x_2^\rho\right)^{1/\rho}$$
$$s.t.\ p_1x_1 + p_2x_2 - M = 0$$

- Lagrangean =

- F.o.c.:

- Solution:

$$\begin{aligned} x_1^* &= \frac{M}{p_1\left(1+\left(\frac{\alpha}{\beta}\right)^{\frac{1}{\rho-1}}\left(\frac{p_2}{p_1}\right)^{\frac{\rho}{\rho-1}}\right)} \\ x_2^* &= \frac{M}{p_2\left(1+\left(\frac{\beta}{\alpha}\right)^{\frac{1}{\rho-1}}\left(\frac{p_1}{p_2}\right)^{\frac{\rho}{\rho-1}}\right)} \end{aligned}$$

- Special case 1: $\rho = 0$ (Cobb-Douglas)

$$
\begin{aligned}
x_1^* &= \frac{\alpha}{\alpha+\beta}\frac{M}{p_1} \\
x_2^* &= \frac{\beta}{\alpha+\beta}\frac{M}{p_2}
\end{aligned}
$$

- Special case 1: $\rho \to 1^-$ (Perfect Substitutes)

$$
\begin{aligned}
x_1^* &= \begin{cases} 0 & \text{if } \; p_1/p_2 \geq \alpha/\beta \\ M/p_1 & \text{if } \; p_1/p_2 < \alpha/\beta \end{cases} \\
x_2^* &= \begin{cases} M/p_2 & \text{if } \; p_1/p_2 \geq \alpha/\beta \\ 0 & \text{if } \; p_1/p_2 < \alpha/\beta \end{cases}
\end{aligned}
$$

- Special case 1: $\rho \to -\infty$ (Perfect Complements)

$$x_1^* = \frac{M}{p_1 + p_2} = x_2^*$$

- Parameter ρ indicates substition pattern between goods:
  - $\rho > 0$ –> Goods are (net) substitutes
  - $\rho < 0$ –> Goods are (net) complements

3 Utility maximization – tricky cases

1. Non-convex preferences. Example:

2. Example with CES utility function.

$$\max_{x_1, x_2} \left(\alpha x_1^{\rho} + \beta x_2^{\rho}\right)^{1/\rho}$$
$$s.t.\ p_1 x_1 + p_2 x_2 - M = 0$$

- With $\rho > 1$ the interior solution is a minimum!

- Draw indifference curves for $\rho = 1$ (boundary case) and $\rho = 2$

- Can also check using second order conditions

2. Solution does not satisfy $x_1^* > 0$ or $x_2^* > 0$. Example:

$$\max x_1 * (x_2 + 5)$$
$$s.t.\ p_1 x_1 + p_2 x_2 = M$$

- In this case consider corner conditions: what happens for $x_1^* = 0$? And $x_2^* = 0$?

3. Multiplicity of solutions. Example:

- Convex preferences that are not strictly convex

4 Indirect utility function

- Nicholson, Ch. 4, pp. 128-130

- Define the indirect utility $v(\mathbf{p}, M) \equiv u(\mathbf{x}^*(\mathbf{p}, M))$, with $\mathbf{p}$ vector of prices and $\mathbf{x}^*$ vector of optimal solutions.

- $v(\mathbf{p}, M)$ is the utility at the optimimum for prices $\mathbf{p}$ and income M

- Some comparative statics: $\partial v(\mathbf{p}, M)/\partial M =$?

- Hint: Use Envelope Theorem on Lagrangean function

- What is the sign of λ?

- $\lambda = u'_{x_i}/p > 0$

- $\partial v(\mathbf{p}, M)/\partial p_i =$?

- Properties:

  - Indirect utility is always increasing in income M

  - Indirect utility is always decreasing in the price p_i

5 Next Class

- Comparative Statics:
  - with respect to price
  - with respect to income