Outline

1. Optimization with 1 variable

2. Multivariate optimization

3. Comparative Statics

4. Implicit function theorem
1 Optimization with 1 variable

- Nicholson, Ch.2, pp. 20-25

- Example. Function $y = -x^2$

- What is the maximum?

- Maximum is at 0

- General method?
• Sure! Use derivatives

• Derivative is slope of the function at a point:
  \[
  \frac{\partial f(x)}{\partial x} = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h}
  \]

• **Necessary condition for maximum** \( x^* \) is
  \[
  \frac{\partial f(x^*)}{\partial x} = 0 \quad (1)
  \]

• Try with \( y = -x^2 \).

  • \[ \frac{\partial f(x)}{\partial x} = \]
  \[ = 0 \implies x^* = \]
• Does this guarantee a maximum? No!

• Consider the function \( y = x^3 \)

\[
\frac{\partial f(x)}{\partial x} = 0 \implies x^* = \]

• Plot \( y = x^3 \).
• **Sufficient condition for a (local) maximum:**

\[
\frac{\partial f(x^*)}{\partial x} = 0 \text{ and } \frac{\partial^2 f(x)}{\partial^2 x} \bigg|_{x^*} < 0 \quad (2)
\]

• Proof: At a maximum, \(f(x^* + h) - f(x^*) < 0\) for all \(h\).

• **Taylor Rule:**

\[
f(x^* + h) - f(x^*) = \frac{\partial f(x^*)}{\partial x} h + \frac{1}{2} \frac{\partial^2 f(x^*)}{\partial^2 x} h^2 + \text{higher order terms.}
\]

• Notice: \(\frac{\partial f(x^*)}{\partial x} = 0\).

• \(f(x^* + h) - f(x^*) < 0\) for all \(h \implies \frac{\partial^2 f(x^*)}{\partial^2 x} h^2 < 0 \implies \frac{\partial^2 f(x^*)}{\partial^2 x} < 0\)

• Careful: Maximum may not exist: \(y = \exp(x)\)
• Tricky examples:

  – *Minimum.* \( y = x^2 \)

  – *No maximum.* \( y = \exp(x) \) for \( x \in (-\infty, +\infty) \)

  – *Corner solution.* \( y = x \) for \( x \in [0, 1] \)
2 Multivariate optimization

- Nicholson, Ch.2, pp. 26-31 and 33-35

- Function from $\mathbb{R}^n$ to $\mathbb{R}$: $y = f(x_1, x_2, \ldots, x_n)$

- Partial derivative with respect to $x_i$:

$$\frac{\partial f(x_1, \ldots, x_n)}{\partial x_i} = \lim_{h \to 0} \frac{f(x_1, \ldots, x_i + h, \ldots x_n) - f(x_1, \ldots, x_i, \ldots x_n)}{h}$$

- Slope along dimension $i$

- Total differential:

$$df = \frac{\partial f(x)}{\partial x_1}dx_1 + \frac{\partial f(x)}{\partial x_2}dx_2 + \ldots + \frac{\partial f(x)}{\partial x_n}dx_n$$
• One important economic example

• Example 1: Partial derivatives of $y = f(L, K) = L^{0.5}K^{0.5}$

• $f'_L =
  \text{(marginal productivity of labor)}$

• $f'_K =
  \text{(marginal productivity of capital)}$

• $f''_{L,K} =$
Maximization over an open set (like $R$)

- **Necessary condition for maximum** $x^*$ is

\[
\frac{\partial f(x^*)}{\partial x_i} = 0 \ \forall i \quad (3)
\]

or in vectorial form

\[
\nabla f(x) = 0
\]

- These are commonly referred to as first order conditions (f.o.c.)

- **Sufficient conditions?** Define Hessian matrix $H$:

\[
H = \begin{pmatrix}
    f''_{x_1,x_1} & f''_{x_1,x_2} & \cdots & f''_{x_1,x_n} \\
    \vdots & \vdots & \ddots & \vdots \\
    f''_{x_n,x_1} & f''_{x_n,x_2} & \cdots & f''_{x_n,x_n}
\end{pmatrix}
\]
• Subdeterminant $|H|_i$ of Matrix $H$ is defined as the determinant of submatrix formed by first $i$ rows and first $i$ columns of matrix $H$.

• Examples.

  – $|H|_1$ is determinant of $f''_{x_1,x_1}$, that is, $f''_{x_1,x_1}$

  – $|H|_2$ is determinant of

    $$H = \begin{pmatrix}
    f''_{x_1,x_1} & f''_{x_1,x_2} \\
    f''_{x_2,x_1} & f''_{x_2,x_2}
    \end{pmatrix}$$

• **Sufficient condition for maximum $x^*$**.

  1. $x^*$ must satisfy first order conditions;

  2. Subdeterminants of matrix $H$ must have alternating signs, with subdeterminant of $H_1$ negative.
• Case with $n = 2$

• Condition 2 reduces to $f''_{x_1,x_1} < 0$ and $f''_{x_1,x_1}f''_{x_2,x_2} - (f''_{x_1,x_2})^2 > 0$.

• Example 2: $h(x_1, x_2) = p_1 x_1^2 + p_2 x_2^2 - 2x_1 - 5x_2$

• First order condition w/ respect to $x_1$?

• First order condition w/ respect to $x_2$?

• $x_1^*, x_2^* =$

• For which $p_1, p_2$ is it a maximum?

• For which $p_1, p_2$ is it a minimum?
3 Comparative statics

- Economics is all about ‘comparative statics’

- What happens to optimal economic choices if we change one parameter?

- Example: Car production. Consumer:
  1. Car purchase and increase in oil price
  2. Car purchase and increase in income

- Producer:
  1. Car production and minimum wage increase
  2. Car production and decrease in tariff on Japanese cars

- Next two sections
4 Implicit function theorem

- Implicit function: Ch. 2, pp. 31-32

- Consider function $x_2 = g(x_1, p)$

- Can rewrite as $x_2 - g(x_1, p) = 0$

- **Implicit function** has form: $h(x_2, x_1, p) = 0$

- Often we need to go from implicit to explicit function

- Example 3: $1 - x_1 * x_2 - e^{x_2} = 0$.

- Write $x_1$ as function of $x_2$:

- Write $x_2$ as function of $x_1$:
• **Univariate implicit function theorem (Dini):** Consider an equation \( f(p, x) = 0 \), and a point \((p_0, x_0)\) solution of the equation. Assume:

1. \( f \) continuously differentiable in a neighbourhood of \((p_0, x_0)\);

2. \( f_x(p_0, x_0) \neq 0 \).

• Then:

1. There is one and only function \( x = g(p) \) defined in a neighbourhood of \( p_0 \) that satisfies \( f(p, g(p)) = 0 \) and \( g(p_0) = x_0 \);

2. The derivative of \( g(p) \) is

\[
g'(p) = -\frac{f'_p(p, g(p))}{f'_x(p, g(p))}
\]
• Example 3 (continued): \(1 - x_1 \cdot x_2 - e^{x_2} = 0\)

• Find derivative of \(x_2 = g(x_1)\) implicitly defined for \((x_1, x_2) = (1, 0)\)

• Assumptions:
  1. Satisfied?
  2. Satisfied?

• Compute derivative
• Multivariate implicit function theorem (Dini):
  Consider a set of equations \( (f_1(p_1, \ldots, p_n; x_1, \ldots, x_s) = 0; \ldots; f_s(p_1, \ldots, p_n; x_1, \ldots, x_s) = 0) \), and a point \((p_0, x_0)\) solution of the equation. Assume:

  1. \( f_1, \ldots, f_s \) continuously differentiable in a neighbourhood of \((p_0, x_0)\);

  2. The following Jakobian matrix \( \frac{\partial f}{\partial x} \) evaluated at \((p_0, x_0)\) has determinant different from 0:

\[
\frac{\partial f}{\partial x} = \begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_s} \\
\cdots & \cdots & \cdots \\
\frac{\partial f_s}{\partial x_1} & \cdots & \frac{\partial f_s}{\partial x_s}
\end{pmatrix}
\]
Then:

1. There is one and only set of functions $x = g(p)$ defined in a neighbourhood of $p_0$ that satisfy $f(p, g(p)) = 0$ and $g(p_0) = x_0$;

2. The partial derivative of $x_i$ with respect to $p_k$ is

$$
\frac{\partial g_i}{\partial p_k} = -\frac{\det \left( \frac{\partial (f_1, \ldots, f_s)}{\partial (x_1, \ldots x_{i-1}, p_k, x_{i+1}, \ldots, x_s)} \right)}{\det \left( \frac{\partial f}{\partial x} \right)}
$$
• Example 2 (continued): Max \( h(x_1, x_2) = p_1 \cdot x_1^2 + p_2 \cdot x_2^2 - 2x_1 - 5x_2 \)

• f.o.c. \( x_1 : 2p_1 \cdot x_1 - 2 = 0 = f_1(p,x) \)

• f.o.c. \( x_2 : 2p_2 \cdot x_2 - 5 = 0 = f_2(p,x) \)

• Comparative statics of \( x_1^* \) with respect to \( p_1 \)?

• First compute \( \text{det} \left( \frac{\partial f}{\partial x} \right) \)

\[
\begin{pmatrix}
\frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2}
\end{pmatrix} = \begin{pmatrix}
\end{pmatrix}
\]
• Then compute $\det \left( \frac{\partial(f_1, \ldots, f_s)}{\partial(x_1, \ldots, x_i-1, p_k, x_i+1, \ldots, x_s)} \right) \left( \begin{array}{cc}
\frac{\partial f_1}{\partial p_1} & \frac{\partial f_1}{\partial x_2} \\
\frac{\partial f_2}{\partial p_1} & \frac{\partial f_2}{\partial x_2} 
\end{array} \right) = \left( \ldots \right)$

• Finally, $\frac{\partial x_1}{\partial p_1} =$

• Why did you compute $\det \left( \frac{\partial f}{\partial x} \right)$ already?
5 Next Class

- Next class:
  - Envelope Theorem
  - Convexity and Concavity
  - Constrained Maximization
  - Envelope Theorem II

- Going toward:
  - Preferences
  - Utility Maximization (where we get to apply maximization techniques the first time)