Field Examination: Econometrics

Department of Economics
University of California, Berkeley

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Instructions: You have 180 minutes to answer **THREE out of the following four questions.** Please make your answers elegant, that is, clear, concise, and, above all, correct. Good luck!

QUESTION 1. Suppose a sample of N i.i.d. observations on a scalar dependent variable y_i and p-dimensional vector of (non-constant) regressors x_i satisfies a linear model

$$y_i = x_i'\beta_0 + \varepsilon_i,$$

where the slope coefficients β_0 are unknown, and the unobservable error term ε_i is statistically independent of the regressors x_i, and the regressors x_i and errors ε_i are bounded with probability one (so all moments exist). Consider an estimator of the slope coefficient vector β_0 that is defined to minimize the average of a convex function of differences in dependent variables $y_i - y_j$ and corresponding differences in regression functions $(x_i - x_j)'\beta$ across all distinct pairs of observations; that is,

$$\begin{aligned} \hat{\beta} &\equiv \arg\min_{\beta \in R^p} S_n(\beta), \\ S_n(\beta) &\equiv \binom{N}{2}^{-1} \sum_{i=1}^{N-1} \sum_{j=i+1}^{N} \rho\left((y_i - y_j) - (x_i - x_j)'\beta\right), \end{aligned}$$

where $\rho(\cdot)$ is an even function ($\rho(u) = \rho(|u|)$) that is smooth and has second derivative bounded away from zero ($\rho''(u) \geq c_0 > 0$).

(a) Under what additional condition(s) will $\hat{\beta}$ be consistent for β_0? Sketch a proof of consistency under the assumptions given above and the additional condition(s) you impose.

(b) Derive an expression for the asymptotic distribution of $\hat{\beta}$. Simplify as much as you can.

(c) Construct a consistent estimator of the asymptotic covariance matrix of $\hat{\beta}$, and give a brief argument for its consistency.

(d) Suppose that ε_i is *not* independent of x_i. Derive an expression for the asymptotic distribution of $\hat{\beta} - \beta^*$, where $\beta^* \equiv p\lim \hat{\beta}$ (assumed to exist).

QUESTION 2. Suppose $X \sim P_0$ with $X \in \mathbb{R}$ and $(X_i)_{i=1}^n$ are IID copies of X. Suppose that P_0 is indexed by $\theta_0 \in Interior\{\Theta\} \subseteq \mathbb{R}^d$ and that θ_0 is such that

$$E_{P_0}[m(X,\theta_0)] > E_{P_0}[m(X,\theta)], \qquad \theta \neq \theta_0,$$

where $m : \mathbb{R}^{1+d} \to \mathbb{R}$ and

(i) Θ is compact.

(ii) $\theta \mapsto m(X,\theta)$ is twice continuously differentiable, a.s.-P_0.

(iii) Let $\theta \mapsto H(\theta) \equiv E_{P_0}[\nabla_{\theta\theta} m(X,\theta)] \in \mathbb{R}^{d\times d}$. H is continuous, $H(\theta_0)$ is non-singular matrix, and there exists a $\delta > 0$ such that

$$\sup_{\theta\in\Theta:||\theta-\theta_0||\leq\delta} ||n^{-1}\sum_{i=1}^n \nabla_{\theta\theta} m(X_i,\theta) - H(\theta)|| = o_{P_0}(1).$$

(iv) $E_{P_0}[\sup_{\theta\in\Theta} ||\nabla_\theta m(X,\theta)||] \leq C < \infty$ and $V(\theta_0) \equiv E_{P_0}[\nabla_\theta m(X,\theta_0)\nabla_\theta m(X,\theta_0)']$ is a positive definite finite matrix.

Please answer the following questions (each has equal weight). If you think more regularity conditions are need, feel free to impose them but you will be penalized if the conditions are redundant/not needed.

(a) Define the M-estimator for θ_0.
(b) Show that the M-estimator, $\hat{\theta}$, is consistent.
(c) Show that $\sqrt{n}(\hat{\theta} - \theta_0) \Rightarrow N(0, H(\theta_0)^{-1}V(\theta_0)H(\theta_0)^{-1})$.
(d) Suppose you want to test whether the first $k < d$ elements of θ are zero. Construct that Wald statistic for this question and derive the asymptotic distribution under the null, $\theta_{0,1} = ... = \theta_{0,k} = 0$. **Hint:** If $X \sim N(0, I_d)$ (where I_d is the $d \times d$ identity matrix) and A is $d \times d$ idempotent matrix with rank $d - k$ then the quadratic form $X'AX$ is chi-squared distributed with $d - k$ degrees of freedom.
(e) For the same test. Construct the "pseudo likelihood ratio (PLR)",

$$2n(\sup_{\theta\in\Theta} n^{-1}\sum_{i=1}^n m(X_i,\theta) - \sup_{\theta\in\Theta:\theta_{[1:k]}=0} n^{-1}\sum_{i=1}^n m(X_i,\theta)),$$

where $\theta_{[1:k]} = (\theta_1, ..., \theta_k)$.

(e.1) Go as far as you can characterizing the asymptotic distribution under the null, $\theta_{0,1} = ... = \theta_{0,k} = 0$. Is it chi-square?

(e.2) Suppose that $m(x,\theta) = \log p_\theta(x)$ where p_θ is a pdf and p_{θ_0} is the true pdf. Revisit your answer for (e.1) for this case: is it chi-square? (If "yes", prove it; if "no", explain why not).

(f) Compare both tests. Which one would you choose and why? (An informal discussion will suffice).

QUESTION 3. Suppose $\{y_t : 1 \leq t \leq T\}$ is an observed time series generated by the model

$$y_t = \mu + u_t, \qquad u_t = \rho u_{t-1} + \varepsilon_t, \qquad t = 1, \ldots, T,$$

where $u_0 = 0$ and $\varepsilon_t \sim i.i.d.\ \mathcal{N}(0,1)$, while $\mu \in \mathbb{R}$ and $\rho \in (-1,1)$ are (possibly) unknown parameters.

(a) Find the log likelihood function $\mathcal{L}(\mu, \rho)$ and, for $r \in (-1,1)$, derive $\hat{\mu}(r) = \arg\max_{\mu} \mathcal{L}(\mu, r)$, the maximum likelihood estimator of μ when ρ is assumed to equal r.

(b) Find the limiting distribution (after appropriate centering and rescaling) of the "oracle" estimator $\hat{\mu}(\rho)$.

(c) Give conditions on $\hat{\rho}$ under which $\hat{\mu}(\hat{\rho})$ asymptotically equivalent to $\hat{\mu}(\rho)$.

(d) Does $\hat{\rho} = 0$ satisfy the condition derived in (c)? If not, determine whether $\hat{\mu}(0)$ is asymptotically equivalent to $\hat{\mu}(\rho)$.

QUESTION 4. Suppose $\left\{(y_t, x_t)' : 1 \leq t \leq T\right\}$ is an observed time series generated by the cointegrated system

$$y_t = \theta_0 x_t + u_t,$$

where

$$\begin{pmatrix} u_t \\ \Delta x_t \end{pmatrix} \sim i.i.d.\ \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)$$

with initial condition $x_0 = 0$.

It can be shown that

$$\begin{pmatrix} T^{-1/2} x_{\lfloor T \cdot \rfloor} \\ T^{-1} \sum_{t=1}^{T} x_t u_t \\ T^{-3/2} \sum_{t=1}^{T} x_t^2 u_t \end{pmatrix} \to_d \begin{pmatrix} B_x(\cdot) \\ \int_0^1 B_x(r)\, dB_y(r) \\ \int_0^1 B_x(r)^2\, dB_y(r) \end{pmatrix},$$

where B_x and B_y are independent Wiener processes.

Let $z_t = (y_t, x_t)'$ and define the function

$$h_T(z_t, \theta) = \begin{pmatrix} x_t/\sqrt{T} \\ x_t^2/T \end{pmatrix} (y_t - \theta x_t).$$

(a) Show that $\Theta_T = \{\theta_0\}$, where $\Theta_T = \left\{\theta : \sum_{t=1}^{T} E\left[h(z_t, \theta)\right] = 0\right\}$.

Let

$$\hat{\theta}_W = \arg\min_\theta g_T(\theta)' W g_T(\theta), \qquad g_T(\theta) = \frac{1}{T} \sum_{t=1}^{T} h_T(z_t, \theta),$$

where W is a symmetric, positive definite 2×2 matrix.

(b) It can be shown that

$$T\left(\hat{\theta}_W - \theta_0\right) \to_d \int_0^1 B_W(r)\, dB_y(r),$$

where B_W is some functional of B_x and W. Verify this claim and express B_W in terms of B_x and W.

Let $\omega_W^2 = \int_0^1 B_W(r)^2\, dr$.

(c) Find W^*, a value of W for which ω_W^2 is minimal, and express $\omega_{W^*}^2$ in terms of B_x.

(d) Propose a feasible estimator $\hat{\theta}$ satisfying

$$T\left(\hat{\theta} - \theta_0\right) \to_d \int_0^1 B_{W^*}(r)\, dB_y(r).$$