Question 1

Suppose \( \{y_t : 1 \leq t \leq T\} \) is an observed time series generated by the model

\[
y_t = \delta t + u_t, \quad u_t = \rho u_{t-1} + \varepsilon_t, \quad t = 1, \ldots, T,
\]

where \( u_0 = 0 \) and \( \varepsilon_t \sim i.i.d. \mathcal{N}(0, 1) \), while \( \delta \) is a parameter of interest and \( \rho \in (-1, 1) \) is a (possibly) unknown nuisance parameter.

(a) Find the log likelihood function \( L(\delta, \rho) \) and, for \( r \in (-1, 1) \), derive \( \hat{\delta}(r) = \arg \max_{\delta} L(\delta, r) \), the maximum likelihood estimator of \( \delta \) when \( \rho \) is assumed to equal \( r \).

(b) Find the limiting distribution (after appropriate centering and rescaling) of the “oracle” estimator \( \hat{\delta}(\rho) \).

(c) Give conditions on \( \hat{\rho} \) under which \( \hat{\delta}(\hat{\rho}) \) asymptotically equivalent to \( \hat{\delta}(\rho) \).
(d) Does $\hat{\rho} = 0$ satisfy the condition derived in (c)? If not, determine whether $\delta(0)$ is asymptotically equivalent to $\hat{\delta}(\rho)$.

**Question 2**

Suppose that economic theory suggests that a latent dependent variable $y_i$ satisfies a classical linear model

$$y_i^* = x_i'\beta_0 + \epsilon_i$$

but that you do not observe $y_i^*$ over its entire range. Instead, you observe a random sample of size $n$ of $(y_i, x_i)$ where $y_i = \tau(y_i^*)$, where

$$\tau(y^*) = \begin{cases} \max\{0, y^*\} & \text{if } y^* \leq 10 \\ -1 & \text{if } y^* > 10 \end{cases}$$

1. Assuming that $\epsilon_i$ is normally distributed with zero mean and unknown variance $\sigma_0^2$, and is independent of $x_i$; derive the form of the average log-likelihood function for the unknown parameters of this problem and the form of the asymptotic distribution of the corresponding maximum likelihood estimator. Go as far as you can characterizing the asymptotic variance.

2. Suppose that the parametric form of the error distribution is unknown. Find a $\sqrt{n}$-consistent estimator of $\beta_0$, imposing a suitable stochastic restriction on the conditional distribution of $\epsilon_i$ given $x_i$, and without imposing a scale normalization on $\beta_0$. If possible, give an expression for the asymptotic distribution of your estimator.

3. Now suppose that $y_i^*$ is never observed, but only the range that it falls into is observed. More specifically,

$$\tau(y^*) = \begin{cases} 0 & \text{if } y^* \leq 10 \\ -1 & \text{if } y^* > 10 \end{cases}$$

Describe an alternative consistent estimator of $\beta_0$ under a semiparametric restriction on the conditional distribution of the errors given the regressors. Is a scale normalization on $\beta_0$ needed, or are all the components of $\beta_0$ (including the scale) identifiable under your restriction?
Question 3

Suppose \( \hat{Q}_n(\theta) \) is a measurable criterion which is twice continuously differentiable a.s. and

\[
\sup_{\theta \in \Theta} |H^{-1/2} \nabla_{\theta \theta} \hat{Q}_n(\theta) H^{-1/2} - I| = o_P(1)
\]

with \( H \) is symmetric and positive definite (thus non-singular), and \( \Theta \subseteq \mathbb{R}^d \).

Suppose that \( \theta_0 \in \Theta \) is the true parameter, and \( \hat{\theta}_n \in \text{int}(\Theta) \) is measurable and

\[
\hat{Q}_n(\hat{\theta}_n) \leq \hat{Q}_n(\theta), \quad \forall \theta \in \Theta.
\]

Finally, suppose that \( \hat{\theta}_n = \theta_0 + O_P(n^{-1/2}) \).

1. Show that for any \( \theta \in \Theta \) and \( t \in \Theta \)

\[
\sup_{\theta} \left| \hat{Q}_n(\theta + H^{-1/2} t) - \hat{Q}_n(\theta) - t' H^{-1/2} \nabla_{\theta} \hat{Q}_n(\theta) - \frac{1}{2} t' t \right| = o_P(t' H^{-1/2} t).
\]

2. Using point (1), show that for any \( v \in \Theta \),

\[
\frac{2 \hat{Q}_n(\hat{\theta}_n + t_n v) - \hat{Q}_n(\hat{\theta}_n)}{t_n^2} = v' H v + o_P(1).
\]

with \( t_n \in \mathbb{R} \) and \( t_n = o(1) \).

3. Suppose \( \sqrt{n} \nabla_{\theta} \hat{Q}_n(\theta_0) \Rightarrow N(0, H) \). Using point (1), show that

\[
2n(\hat{Q}_n(\theta_0) - \hat{Q}_n(\hat{\theta}_n)) \Rightarrow \chi^2_d.
\]

4. Suppose that \( H \) is not non-singular; but there exists a \( H_n \) such that:

\[
\sqrt{n} H_n^{-1/2} \nabla_{\theta} \hat{Q}_n(\theta_0) \Rightarrow N(0, I),
\]

\[
\sqrt{n} H_n^{-1/2} (\hat{\theta}_n - \theta_0) = O_P(1), \quad \text{and}
\]

\[
\sup_{\theta} |H_n^{-1/2} \nabla_{\theta \theta} \hat{Q}_n(\theta) H_n^{-1/2} - I| = o_P(1)
\]

Can you extend results (1)-(3) under these assumptions? Do you still obtain asymptotic normality? At root-n rate?
Question 4

Suppose \( \{(y_t, x_t, z_t) : 1 \leq t \leq T\} \) is an observed time series generated by the cointegrated system

\[
y_t = \beta x_t + u_t,
\]

where

\[
\begin{pmatrix}
  u_t^y \\
  \Delta x_t \\
  \Delta z_t
\end{pmatrix} \sim i.i.d. \mathcal{N}
\begin{pmatrix}
  \begin{pmatrix}
    0 \\
    0 \\
    0
  \end{pmatrix},
  \begin{pmatrix}
    1 & 0 & \sigma_{zu} \\
    0 & 1 & \sigma_{zx} \\
    \sigma_{zu} & \sigma_{zx} & 1
  \end{pmatrix}
\end{pmatrix}
\]

with initial conditions \( x_0 = z_0 = 0 \), while \( \beta \) is a parameter of interest and \( \sigma_{zu} \) and \( \sigma_{zx} \) are nuisance parameters. Let \( \hat{\beta}_{IV} = \left( \sum_{t=1}^{T} z_t x_t \right)^{-1} \left( \sum_{t=1}^{T} z_t y_t \right) \) be the IV estimator of \( \beta \) that uses \( z_t \) as an instrument.

Suppose it is known that \( \sigma_{zu} = 0 \), but suppose \( \sigma_{zx} \) is unknown.

(a) Derive the maximum likelihood estimator of \( \beta \) and characterize its limiting distribution (after appropriate centering and rescaling).

(b) Assuming \( \sigma_{zx} \neq 0 \), characterize the limiting distribution (after appropriate centering and rescaling) of \( \hat{\beta}_{IV} \). Is \( \hat{\beta}_{IV} \) consistent if \( \sigma_{zx} = 0 \)?

Suppose it is known that \( \sigma_{zx} = 1/2 \), but suppose \( \sigma_{zu} \) is unknown.

(c) Derive the maximum likelihood estimator of \( \beta \) and characterize its limiting distribution (after appropriate centering and rescaling).

(d) Assuming \( \sigma_{zu} \neq 0 \), characterize the limiting distribution (after appropriate centering and rescaling) of \( \hat{\beta}_{IV} \). Is \( \hat{\beta}_{IV} \) a consistent estimator of \( \beta \)?