Economics 209B
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Repeated Games and Recursive Contracts

Notes on Stochastic Calculus.

Disclaimer: This exposition of stochastic calculus is highly informal. See a book like Karatzas and Shreve for an extensive formal treatment of stochastic calculus, or the appendix of "Dynamic Asset Pricing" by Duffie for a concise exposition, which includes technical details.

Random Processes.

A *random process* is a random function $X : [0,\infty) \to \Re^d$ that unfolds over time.

Example 1. Let $X_s = 0$ for $s \in [0,1)$. For $s \geq 1$, let $X_s = s$ with probability 1/2 and $X_s = -s$ with probability 1/2.

Here $[0,\infty)$ represents the timeline. At time t the path X_s, $s \in [0,t]$ is known, but the values of X after time t may or may not be known. The information that one learns by observing the path X_s, $s \in [0,t]$ is denoted by $\mathcal{F}_t^X$. Formally, $\{\mathcal{F}_t^X\}_{t\geq 0}$ is called the filtration generated by X. In general, it may be desirable to model a setting where one knows more information at time t than the information contained in the path X_s, $s \in [0,t]$ alone. In that case, the information known at time t is typically denoted by $\mathcal{F}_t$.

Brownian Motion.

A *Brownian motion* Z_t, $t \in [0,\infty)$ is a random process such that

1. $Z_0 = 0$
2. $Z_{t+s} - Z_t \sim N(0, t-s)$, independently of $\mathcal{F}_t^Z$.

Question 1: *What is the covariance between* $Z_{t+s} - Z_t$ *and* Z_t?
Answer: 0

Question 2: *What is the covariance between* Z_{t+s} *and* Z_t?
Answer: $\mathrm{Cov}(Z_t, Z_{t+s}) = \mathrm{Cov}(Z_t, Z_t) + \mathrm{Cov}(Z_t, Z_{t+s} - Z_t) = \mathrm{Var}(Z_t) + 0 = t.$

Let us lay out a few properties of the Brownian motion in order to understand it better. First, Brownian motion is a martingale:

Definition. A random process M_t, $t \geq 0$ is a *martingale* if for all s and $t > s$,

$$E[M_t|\mathcal{F}_s] = M_s,$$

where $E[\cdot|\mathcal{F}_s]$ denotes the expectation conditional on all the information known at time s.

Example 2. Let Y_T be a random variable that is revealed at a distant future at time T. Define

$$M_t = E[Y_T|\mathcal{F}_t].$$

Then M_t is a martingale.

Question 3: *Is the process defined in example 1 a martingale?*

Question 4: *Prove that the process defined in example 2 is a martingale.*

To visualize the Brownian motion better, let us see a few properties of its sample paths. First, note that $\mathrm{Var}(Z_{t+s} - Z_t) = s$, so during the time interval of length s the Brownian motion moves by a distance on the order of $\sqrt{s}$ (the standard deviation of $Z_{t+s} - Z_t$. When s is small, $\sqrt{s}$ is disproportionately large compared with s. Therefore, the Brownian motion "wiggles" very wildly on a small scale. Also, it is important that the Brownian motion has continuous sample paths.

Question 5: *Argue that the length of a sample path of the Brownian motion from time 0 until time 1 is infinite.*

Because the Brownian motion wiggles so wildly, if $Z_t = x$, then the Brownian motion passes through x infinitely many times from time t to time $t + \epsilon$.

Stochastic Intergral.

If β_t, $t \geq 0$ is a random process then the stochastic integral

$$M_t = \int_0^t \beta_s dZ_s$$

is also a random process. The value of the stochastic integral can be understood as follows. If we subdivide the interval $[0, t]$ into a fine partition $0 = t_0 < t_1 \ldots < t_N = t$ then approximately

$$\int_0^t \beta_s dZ_s \approx \beta_{t_0}(Z_{t_1} - Z_{t_0}) + \beta_{t_1}(Z_{t_2} - Z_{t_1}) + \ldots + \beta_{t_{N-1}}(Z_{t_N} - Z_{t_{N-1}})$$

for all nice processes β. In other words, the stochastic integral adds up the wiggles in the Brownian motion Z, by multiplying the wiggle at time s with a weight β_s. Therefore,

$$\int_0^t 1\, dZ_s = Z_t,$$

since this stochastic integral adds up all the wiggles of the Brownian motion with weight 1. Similarly,

$$\int_0^t 2\, dZ_s = 2Z_t.$$

Question 6: *Let* $\beta_s = 0$ *for* $s \in [0,1)$, $\beta_s = 1$ *for* $s \in [1,2)$ *and* $\beta_s = 2$ *for* $s \geq 2$. *Define*

$$M_t = \int_0^t \beta_s dZ_s.$$

What is M_3? *What is* $E[M_3|\mathcal{F}_2^Z]$?

It is a fact (given here without a proof) that if process β satisfies appropriate regularity conditions, then

$$M_t = \int_0^t \beta_s \, dZ_s \tag{1}$$

is a martingale. For example, this is true if β is a bounded process.

Another way of writing (1) is

$$M_0 = 0, \quad dM_t = \beta_t \, dZ_t.$$

Question 7: *If we write*

$$dW_t = \mu_t \, dt + \beta_t \, dZ_t, \tag{2}$$

what is another way of writing this?

Answer: $W_t = \int_0^t \mu_s \, ds + \int_0^t \beta_s \, dZ_s$, where the first integral should be interpreted as an ordinary, nonstochastic integral of the sample path of process μ.

Ito's formula.

A diffusion process is a random process of the form

$$dW_t = \mu_t \, dt + \beta_t \, dZ_t,$$

where μ_t is called the drift and β_t is called the volatility. Any process that is representable in this form is a diffusion process. The process in Example 1 is not a diffusion process because it has a jump.

If W_t is a diffusion process and $f : \Re \to \Re$ is a twice continuously differentiable function, then $f(W_t)$ is also a diffusion process represented as

$$df(W_t) = \left(f'(W_t)\mu_t + f''(W_t)\frac{\beta_t^2}{2} \right) dt + f'(W_t)\beta_t dZ_t.$$

The formula above is called the Ito's formula. It is the differentiation formula for stochastic calculus.

Let us interpret all terms of Ito's formula.

Question 8: *Let* w_t *defined by* $dw_t/dt = \mu_t$ *be a deterministic process/function. Show that an application of Ito's formula to this process is ordinary differentiation. This iterprets term* $f'(W_t)\mu_t$ *in Ito's formula.*

Question 9. *Interpret the remaining terms of Ito's formula.*

Term $f''(W_t)\frac{\beta_t^2}{2}$ may seem particularly counterintuitive, because it has no analogue in ordinary calculus.

Sometimes it is convenient to consider a function $f(W, t)$ of two variables. Ito's formula for such functions has an extra term

$$df(W_t, t) = \left(f_w(W_t, t)\mu_t + f_{ww}(W_t, t)\frac{\beta_t^2}{2} + f_t(W_t, t) \right) dt + f_w(W_t, t)\beta_t dZ_t.$$

Question 10. *Let $f(w) = w^2$ and $W_t = t + Z_t$. Using Ito's formula, derive $df(W_t)$.*

Question 11. *Let $f(w, t) = e^{-rt}w$ and $dW_t = \beta_t dZ_t$. Using Ito's formula, derive $df(W_t, t)$.*

Question 12. *Let $W_t = tZ_t$. What are the drift and volatility of W_t?*

Martingale Representation Theorem.

Martingale Representation Theorem. Suppose that M_t is a martingale measurable with respect to $\mathcal{F}^Z$. Then there exists a random process β_t such that

$$dM_t = \beta_t \, dZ_t.$$

This theorem says that whenever M_t is known from the path Z_s, $s \in [0, t]$, it can be represented as a stochastic integral with respect to Z. One curious consequence of this is that any martingale measurable with respect to a Brownian filtration cannot have jumps.

Question 13: *Let $M_t = E[Z_T|\mathcal{F}_t^Z]$. What is the process β in the martingale representation for M_t.*

Question 14: *Let $M_t = \text{Prob}[Z_T \geq 0|\mathcal{F}_t^Z]$. Prove that M_t is a martingale and thus has a martingale representation. Find β_t using Ito's formula.*

Girsanov's Theorem.

Let Z be a standard Brownian motion. One way to view Z is as a collection of sample paths, with a probability measure over sample paths. Girsanov's theorem tells us that if we alter the probability measure by making some paths more likely and other paths less likely, then we can endow Z with any drift. Intuitively, if we wanted Z to have drift 1, we would make paths that go up more likely and paths that go down less likely.

Let P be the original probability measure, and Q be a new measure under which Z has drift θ. Formally, the change of measure is defined via a likelihood ratio process ξ (which is $\mathcal{F}^Z$-measurable). The meaning of ξ_t is the following: any given path Z_s, $s \in [0, t]$ is more likely to arise under Q than under P by a factor of ξ_t. Girsanov's theorem not only tells us that such a change of measure can be performed, but also tells us what ξ is.

Girsanov's Theorem. Let θ be a random process that satisfies appropriate technical conditions and suppose that

$$\xi_0 = 0, \quad d\xi_t = -\xi_t\theta_t dZ_t.$$

Define measure Q by

$$Q[A] = E^P[\xi_t 1_A],$$

for any event A known at time t. Then

$$Z_t^Q = Z_t + \int_0^t \theta_s \, ds$$

is a standard Brownian motion under Q.