Myerson (1981), Optimal Auction Design

N bidders, whose types t_i are independently drawn from densities $f_i : [a_i, b_i] \to R_+$, CDF $F_i : [a_i, b_i] \to [0,1]$. The seller's own type t_0 is non-random and known to all bidders. Each bidder is uncertain about the quality of the object for sale, and would revise her best estimate if she learned estimates of other bidders. If a bidder knew everybody's types, her valuation would be

$$v_i(t) = t_i + \sum_{j \neq i} e_j(t_j).$$

Similarly, if the seller had known the types of the bidders, he would reassess his personal valuation to

$$v_0(t) = t_0 + \sum_{i} e_j(t_j).$$

However, initially each bidder and the seller know only their own estimates and believe that the value estimates of other bidders are random and independent with CDF F_i.

Seller's problem: select an auction mechanism to maximize his expected utility.

The Revelation Principle.

One can imagine very complicated auction designs. However, it turns out that one can limit attention to a special class of mechanisms, called *direct revelation mechanisms*. In such a mechanism the bidders simultaneously and confidentially announce their value estimates to the seller, and the mechanism is designed in such a way that truth-telling is a Bayesian Nash equilibrium. The seller determines who gets the object and how much everybody pays as a function of announced types $t = (t_1, \ldots t_N)$. Such a mechanism can be summarized by a pair of functions $(p_i(t), x_i(t))$ for each bidder, where $p_i(t)$ is the probability that i gets the object and $x_i(t)$ is i's expected payment when t is the vector of announcements.

In this mechanism, the expected utility of bidder i is given by

$$U_i(p, x, t_i) = \int_{T_{-i}} (v_i(t) p_i(t) - x_i(t)) f_{-i}(t_{-i}) dt_{-i}$$

and the expected utility of the seller is given by

$$U_0(p, x) = \int_T \left(v_0(t) \left(1 - \sum_{j \in N} p_j(t) \right) + \sum_{j \in N} x_j(t) \right) f(t)\, dt$$

An auction mechanism is feasible if three conditions hold:

$$\sum_{j \in N} p_j(t) < 1 \quad \text{and} \quad p_i(t) > 0, \qquad \forall i \in N, \quad \forall t \in T. \qquad \text{(probability)}$$

$$U_i(p, x, t_i) > 0, \qquad \forall i \in N, \quad \forall_{t_i} \in [a_i, b_i]. \qquad \text{(IR)}$$

$$U_i(p, x, t_i) > \int_{T_{-i}} (v_i(t) p_i(t_{-i}, s_i) - x_i(t_{-i}, s_i)) f_{-i}(t_{-i})\, dt_{-i} \qquad \text{(IC)}$$

$$\forall i \in N, \quad \forall t_i \in [a_i, b_i], \quad \forall s_i \in [a_i, b_i].$$

Why can we limit attention to direct revelation mechanisms?

LEMMA 1. (THE REVELATION PRINCIPLE.) *Given any feasible auction mechanism, there exists an equivalent feasible direct revelation mechanism which gives to the seller and all bidders the same expected utilities as in the given mechanism.*

In general, one can imagine very complicated auction games, in which each bidder has strategy options Θ_i and the outcome function is given by

$$\hat{p} : \Theta_1 \times \cdots \times \Theta_n \to \mathbb{R}^n \quad \text{and} \quad \hat{x} : \Theta_1 \times \cdots \times \Theta_n \to \mathbb{R}^n$$

For any Bayesian Nash equilibrium $\{\hat{\theta}_i(t_i)\}_{i \in I}$ of such an alternative mechanism, one can construct a payoff-equivalent direct revelation mechanism by letting

$$p(t_1, \ldots, t_n) = \hat{p}(\hat{\theta}_1(t_1), \ldots, \hat{\theta}_n(t_n)),$$

$$x(t_1, \ldots, t_n) = \hat{x}(\hat{\theta}_1(t_1), \ldots, \hat{\theta}_n(t_n)).$$

Effectively, the seller elicits true type from each bidder by promising to play for him strategy $\hat{\theta}_i(t_i)$ in the alternative mechanism $(\hat{p}, \hat{x})$ and deliver the outcome from this alternative mechanism.

The Revenue Equivalence Theorem and Optimal Auctions.

The probability of winning the object by announcing t_i is

$$Q_i(p, t_i) = \int_{T_{-i}} p_i(t) f_{-i}(t_{-i})\, dt_{-i}$$

The following Lemma simplifies feasibility conditions for a direct revelation mechanism.

LEMMA 2. *(p,x) is feasible if and only if the following conditions hold:*

$Q_i(p, t_i)$ is weakly increasing in t_i (monotonicity)

$$U_i(p,x,t_i) = U_i(p,x,a_i) + \int_{a_i}^{t_i} Q_i(p,s_i)ds_i, \quad \forall i \in N, \quad \forall t_i \in [a_i, b_i]; \qquad \text{(IC}'\text{)}$$

$$U_i(p,x,a_i) \geq 0, \qquad \forall i \in N; \qquad \text{(IR}'\text{)}$$

and (probability)

Sketch of Proof. If type t_i pretends to be type s_i, he will make the same expected payment as type s_i and get the object with the same probability as type s_i, but in case he obtains the object, he will value it as t_i, not s_i. Therefore, his expected utility is

$$U_i(p,x,s_i) + (t_i - s_i)Q_i(p,s_i).$$

Since this expression is maximized by choosing $s_i = t_i$, the first order condition implies

$$\frac{\partial}{\partial t_i} U_i(p,x,t_i) - Q(p,t_i) = 0,$$

which implies (IC′). (See Myerson (1981) for an argument that does not rely on differentiability). With (IC′), (IR) implies (IR′). See Myerson (1981) for the converse. QED

The following Lemma characterizes optimal auctions:

LEMMA 3. *Suppose that $p : T \to \mathbb{R}^n$ maximizes*

$$\int_T \left(\sum_{i \in N} \left(t_i - e_i(t_i) - \frac{1 - F_i(t_i)}{f_i(t_i)} - t_0 \right) p_i(t) \right) f(t)\, dt \qquad (4.7)$$

subject to constraints (probability) and (monotonicity). Suppose also that

$$x_i(t) = p_i(t) v_i(t) - \int_{a_i}^{t_i} p_i(t_{-i}, s_i)\, ds_i, \qquad \forall i \in N, \quad \forall t \in T. \qquad (4.8)$$

Then (p, x) represents an optimal auction.

Proof. The total surplus (expected sum of utilities of all buyers and the seller) is

$$\int_T \left(\sum_{i \in N} \left(t_i + \sum_{j \neq i} e_j(t_j) \right) p_i(t) + \left(t_0 + \sum_{i \in N} e_j(t_j) \right) \left(1 - \sum_{i \in N} p_i(t) \right) \right) f(t) dt =$$

$$\int_T \left(\sum_{i \in N} (t_i - e_i(t_i) - t_0) p_i(t) + \left(t_0 + \sum_{i \in N} e_j(t_j) \right) \right) f(t) dt = (*)$$

The seller's share of the total surplus is (*) minus the expected utility that each buyer gets. The expected utility of buyer i is

$$- \int_{a_i}^{b_i} U_i(p, x, t_i) f_i(t_i)\, dt_i = - \int_{a_i}^{b_i} \left(U_i(p, x, a_i) + \int_{a_i}^{t_i} Q_i(p, s_i) ds_i \right) f_i(t_i)\, dt_i$$

$$= - U_i(p, x, a_i) - \int_{a_i}^{b_i} \int_{s_i}^{b_i} f_i(t_i) Q_i(p, s_i)\, dt_i\, ds_i = - U_i(p, x, a_i) - \int_{a_i}^{b_i} (1 - F_i(s_i)) Q_i(p, s_i)\, ds_i$$

$$= - U_i(p, x, a_i) - \int_T (1 - F_i(t_i)) p_i(t) f_{-i}(t_{-i})\, dt.$$

Subtracting each buyer's expected utility from (*) we obtain

$$\int_T \sum_i \left(t_i - e_i(t_i) - t_0 - \frac{1 - F_i(t_i)}{f_i(t_i)} \right) p_i(t) f(t) dt - \sum_i U_i(p, x, a_i) + \int_T \left(t_0 + \sum_{i \in N} e_j(t_j) \right) f(t) dt \quad \text{(x)}$$

The last term does not depend on auction design, and it is optimal to have $U_i(p, x, a_i) = 0$ for all i. Therefore, an optimal auction maximizes (4.7). Finally, if x_i are given by (4.8), then it is easy to check that $U_i(p, x, t_i)$ satisfy $U_i(p, x, a_i) = 0$ and (IC′). We conclude that the auction that satisfies the conditions in the lemma is optimal.

From the simple expression (x) for the seller's expected utility, we get the following result:

COROLLARY (THE REVENUE-EQUIVALENCE THEOREM). *The seller's expected utility from a feasible auction mechanism is completely determined by the probability function p and the numbers $U_i(p, x, a_i)$ for all i.*

That is, once we know who gets the object in each possible situation (as specified by p) and how much expected utility each bidder would get if his value estimate were at its lowest possible level a_i, then the seller's expected utility from the auction does not depend on the payment function x. Thus, for example, the seller must get the same expected utility from any two auction mechanisms which have the properties that (1) the object always goes to the bidder with the highest value estimate above t_o and (2) every bidder would expect zero utility if his value estimate were at its lowest possible level. If the bidders are symmetric and all $e_i = 0$ and $a_i = 0$, then the Dutch auctions and progressive auctions studied in [11] both have these two properties, so Vickrey's equivalence results may be viewed as a corollary of our equation (4.12). However, we shall see that Vickrey's auctions are not in general optimal for the seller.

If the function

$$c_i(t_i) = t_i - e_i(t_i) - \frac{1 - F_i(t_i)}{f_i(t_i)}$$

is strictly increasing, then the design of the optimal auction is easy: give the good to the bidder with the highest $c_i(t_i)$ (or let the seller keep the good if t_0 is bigger than $c_i(t_i)$ for any of the buyers). Let the buyers pay according to (4.8).