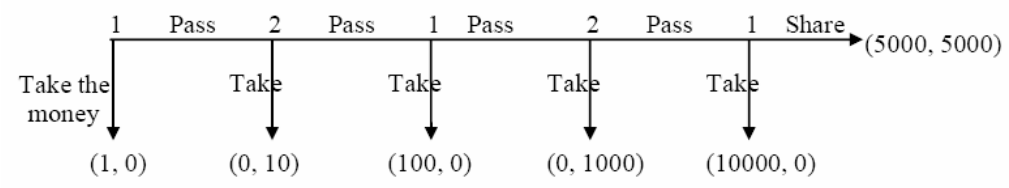
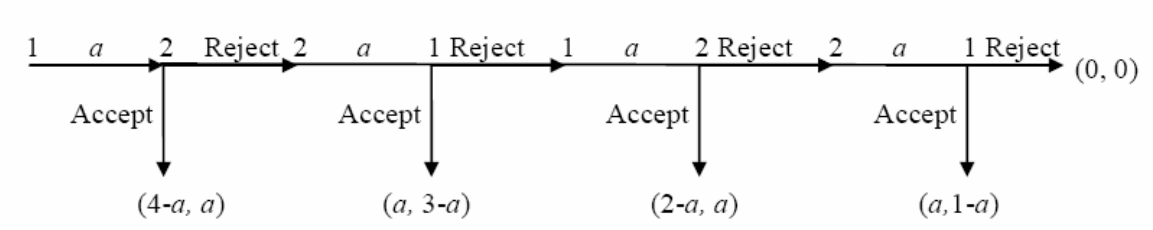
Extensive Form Games, Subgame Perfect Equilibrium, PBE, Sequential Equilibrium

Review

Examples:



Centipede game.



Rubinstein's bargaining game.

Subgame: a node with all its successors, so there are no broken information sets.

Subgame perfect equilibrium: Combination of strategies which form a Nash equilibrium in each subgame.

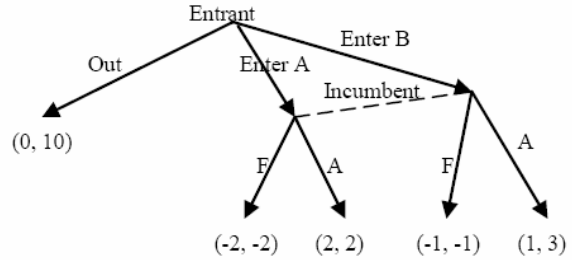
Pure strategy: specifies an action in each node.
Mixed strategy: probability distribution over pure strategies. (In games with perfect recall, this is the same as having a probability distribution over actions in each info set).

	C	D
C	1, 1	-1, 2
D	2, -1	0, 0

→

	C	D
C	1, 1	-1, 2
D	2, -1	0, 0

Twice repeated Prisoners' Dilemma (how many subgames does it have)?



Entrant-incumbent game: find subgame perfect equilibria.

Definition: A *belief* assigns a probability measure to each information set. A combination of strategies and beliefs is a *perfect Bayesian equilibrium (PBE)* if

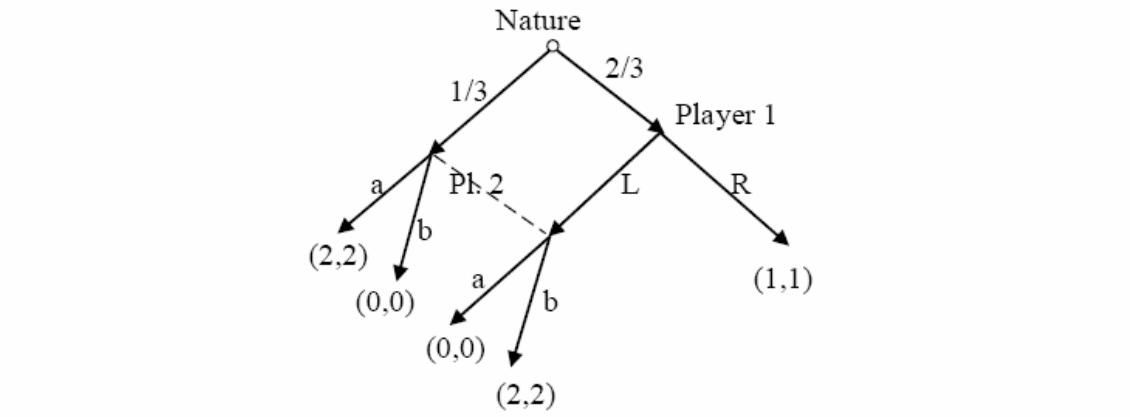
- Beliefs in each information set are formed by Bayes rule from strategies
- The action of each player in each information set is optimal given his belief, and the strategies of all players in the future.

For the entrant-incumbent game above, the incumbent must choose A no matter what his belief is, and the entrant will choose enter A. As a result, the incumbent's belief places probability 1 on the left node, and probability 0 on the right node.

Exercise (from 201A final).

Problem 1. (20 points) In the game below, please find all Perfect Bayesian Equilibria. Please specify all strategies and beliefs clearly in the following format:

	Player 1's strategy	Player 2's strategy	Player 2's belief
Equilibrium 1			
Equilibrium 2			
Equilibrium 3			



Example: education signaling game. There are two types of workers with productivities $x_L < x_H$. Productivity is unobservable to the market, but each worker can choose education level $e \in [0, \infty)$ at cost ec_L or ec_H depending on his type, with $c_H < c_L$. After observing education, many employers simultaneously name wages w_i. The worker chooses the employer i and gets payoff $w_i - ce$, while the employer gets payoff $x - w_i$.

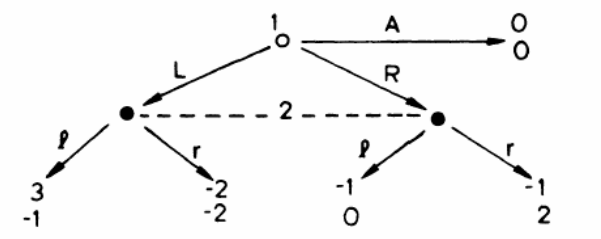
Characterize all PBE.

Kreps and Wilson "Sequential Equilibria."

Introduce a new equilibrium criterion for extensive form games, *sequential equilibrium,* which involves beliefs, and is easier to work with than Selten's criterion of *trembling hand perfection.*

Section 2: Formalism about extensive form games: need to formally represent the order of moves, randomness, information players have when they move, payoffs

(1) finite set of nodes T: initial nodes have no predecessors; each non-initial node has one predecessor; non-terminal nodes X have successors; terminal nodes have successors; no cycles



Example of an extensive-form game.

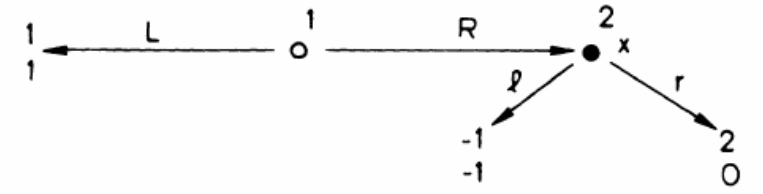
(2) each non-terminal node has the same number of actions as successors, with one action leading to each successor
(3) mapping that assigns a player to each decision node
(4) nodes are partitioned into information sets. Within an information set, each node has the same player, and the same number of actions
Perfect recall: if two nodes are in the same information set of player i, then the same sequence of player i's actions leads to both nodes.
(5) payoff functions u^i that assign the payoff of player i to each terminal node
(6) initial probability distribution over initial nodes
(7) *A strategy* of player *i* defines a probability measure over his actions to each of his information set (this definition works for games with perfect recall)

Section 3: Discussion of Nash Equilibria and Subgame Perfection, and how they are unsatisfactory.

Subgame: a node with all its successors with no broken information sets

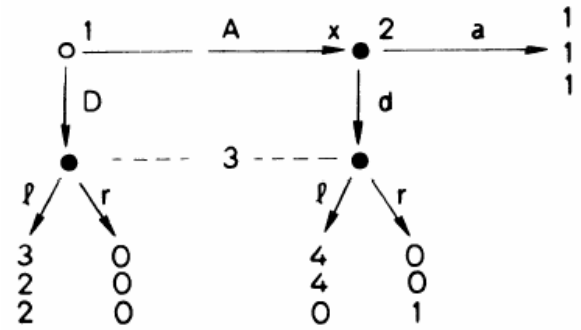
Nash equilibrium: (can define even though we have an extensive form game): a pair of strategies, such that each player maximizes his payoff given the strategies of the opponents.

Subgame Perfect Equilibrium: combination of strategies, which form a Nash equilibrium in each subgame



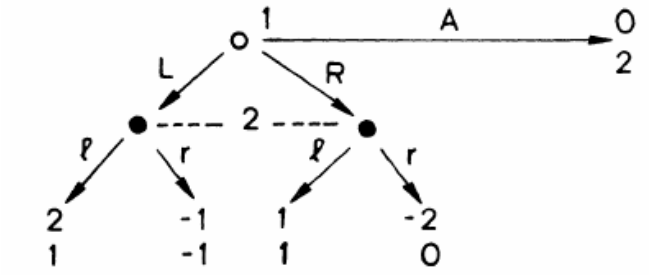
Nash vs. Subgame Perfect Equilibrium.

Subgame perfection has the right idea, but is unsatisfactory for many games with imperfect info.



Subgame Perfection fails: Dla unreasonable equilibrium, but subgame perfect.

Could try to refine the definition, e.g. add that a player whose information set is a singleton cannot improve his payoff conditional upon reaching that point by choosing a different action. (Why only singletons?)



A refined definition would have no bite on Ar.

Because of information sets, need to introduce beliefs in order to be able to compute expected payoffs and refine subgame perfection.

Definition: A *system of beliefs* is defined as a function $\mu : X \to [0, 1]$, such that $\mu(x)$ add up to 1 for x in each information set. An *assessment* is a pair (μ, π).

An assessment is *sequentially rational* if each player's strategy maximizes his payoff from each information set, given his belief there.

$$E^{\mu,\pi}\left[u^{\iota(h)}(z) \mid h\right] \geq E^{\mu,\bar{\pi}}\left[u^{\iota(h)}(z) \mid h\right]$$

$$\text{for all } \bar{\pi} \text{ such that } \bar{\pi}^j = \pi^j \text{ for } j \neq \iota(h).$$

In other words, *sequentially rational* means strategies are good given beliefs.

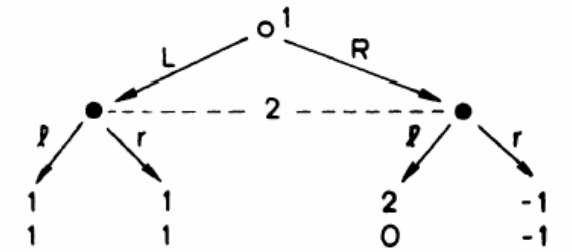
Assessment is *consistent* means <u>beliefs are good given strategies</u> (of course, this has to be formalized).

DEFINITION: An assessment (μ, π) is *consistent* if $(\mu, \pi) = \lim_{n\to\infty}(\mu_n, \pi_n)$ for some sequence $\{(\mu_n, \pi_n)\} \subseteq \Psi^0$. The set of consistent assessments is denoted by Ψ. (That is, Ψ is the closure of Ψ^0.)

A *sequential equilibrium* is an assessment (μ, π) that is both consistent and sequentially rational.

From nearly the end of the paper, here is a very similar refinement proposed by Selten:

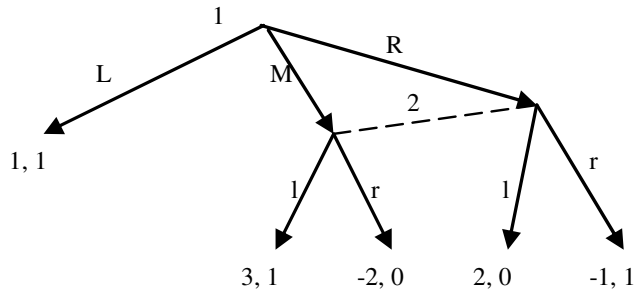
Definition: *(μ, π)* is a *(trembling hand) perfect equilibrium* if $(\mu, \pi) = \lim_{n\to\infty}(\mu_n, \pi_n)$ and π is a best response to π_n for all n.



Lr is a sequential equilibrium that is not perfect.

So far, we've discussed Sections 2 and 3 (and something from one of the sections at the end). Now we take a detour to do exercises and understand sequential equilibrium in greater detail. After exercises, we come back and continue with Section 4.

<u>Detour:</u>

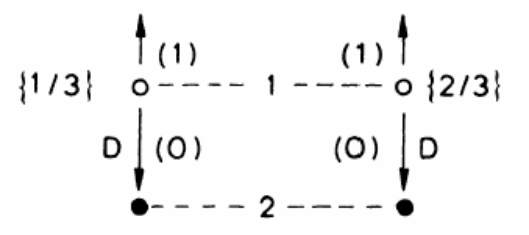


Find *sequential equilibria* of this game.

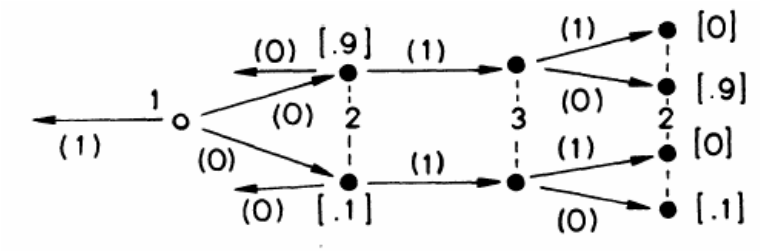
<u>Education signaling game:</u> Is it true that each separating and pooling Perfect Bayesian Equilibrium is also sequential?

Back to the paper.

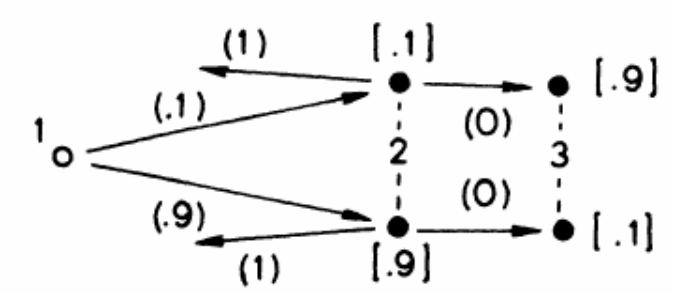
Now, come back to the question: justify why the definition of consistent beliefs is a good one? Answer: because it captures on a number of examples what we think is intuitive. Because it "beats" some other definitions that sound reasonable, like *lexicographic consistency* (there is a sequence of hypotheses about how the game is played $\pi(1)$, $\pi(2)$ … such that the next hypothesis is adapted when the previous once is shown to be wrong). (This is the objective of Section 4.)



It is reasonable for pl. 2 to have beliefs (1/3, 2/3), and consistency gives that (how about lexicographic consistency).



Player 2's beliefs in the last information set are unreasonable, but lexicographically consistent (why?). Yet, they are not consistent.



Again, pl. 3's beliefs are lexicographically consistent, but not consistent.

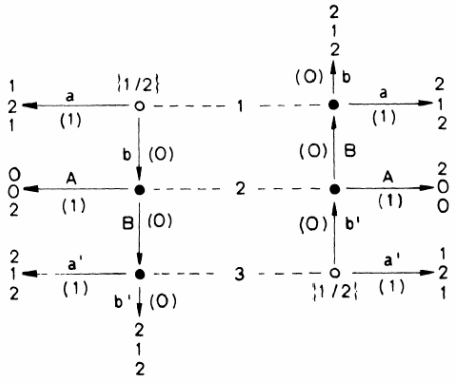
Intuitively, consistency is like lexicographic consistency with the additional restriction that $\pi(n+1)$ should be a minimal change from $\pi(n)$ (if possible with only one player changing his strategy, as opposed to more).

Properties of Sequential Equilibria:

PROPOSITION 1: *For every extensive game, there exists at least one sequential equilibrium.*

PROPOSITION 3: *If* (μ, π) *is a sequential equilibrium, then* π *is a subgame perfect Nash equilibrium.*

In addition, it is shown that sequential equilibrium has some desirable properties, for example it is *extended subgame perfect* (we do not need to know what it is).



Sequential vs. extended subgame perfect.

The paper also discusses topological properties of the sequential equilibrium. We will not discuss them here, but several pictures from the paper might be useful as they give us a sense of what to expect if we try to find sequential equilibria:

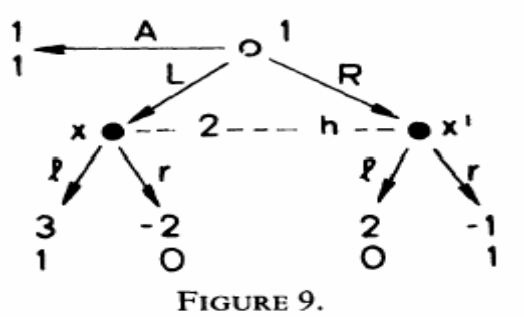


FIGURE 9.

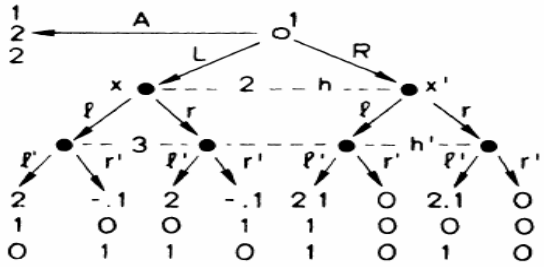


FIGURE 11.

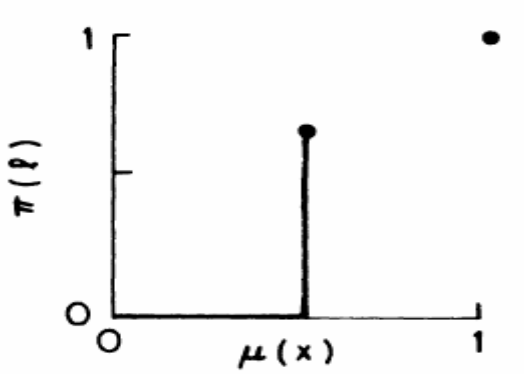


FIGURE 10.

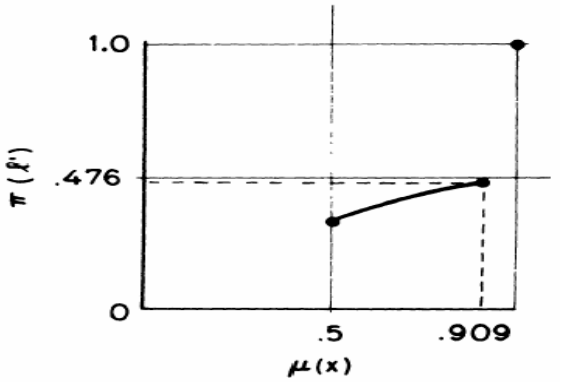
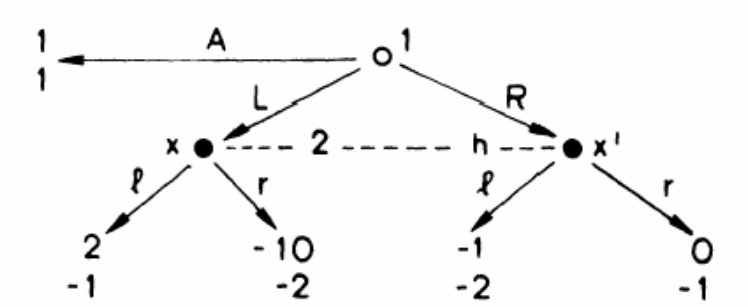


FIGURE 12.

Sequential Equilibrium Does not Capture Some Reasoning.



Ar is one sequential equilibrium, which could be thought unreasonable.