Notes on Fudenberg, Levine and Maskin (1994)

Notation (different from APS).

Stage game:
n players
$a_i \in A_i$: action of player i (from a finite set of actions),
$a \in A = \times_{i=1}^{n} A_i$: action profile
$y \in Y$: signal (from a finite set of signals)
$r_i(a_i, y)$: realized payoff
Expected payoff: $g_i(a) = \sum_{y \in Y} \pi(y \mid a) r_i(a_i, y)$
$\alpha_i \in \Delta(A_i)$: mixed action; π and g_i can be extended to mixed actions

Set of feasible payoffs:

$$V = \{g(a) = (g_1(a), \ldots g_n(a)) \mid a \in A\}$$

Mixed strategy minimax

$$\underline{v}_i = \min_{\alpha_{-i}} \max_{a_i} g_i(a_i, \alpha_{-i})$$

Feasible and individually rational payoffs:

$$V^* = \{v \in V \mid v_i \geq \underline{v}_i \text{ for all } i\}$$

Repeated with discount factor δ:

Periods: t = 0, 1, 2…
Public history: $h^t = (y^0, y^1, \ldots y^t)$.
Private history: $h_i^t = (a_i^0, a_i^1, \ldots a_i^t)$.
A strategy is a sequence of functions $\{\sigma_i^t\}_{t=0}^{\infty}$, where σ_i^t maps (h^{t-1}, h_i^{t-1}) to $\Delta(A_i)$.
Average discounted payoff:

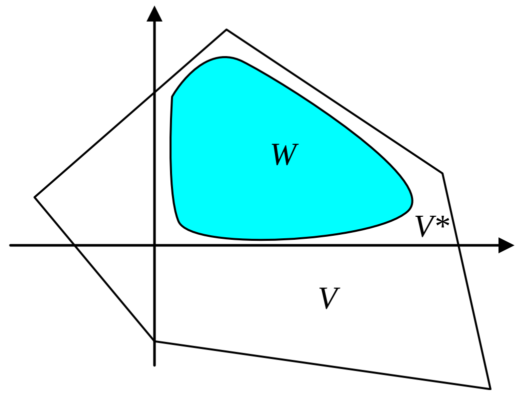
$$(1-\delta)\sum_{t=0}^{\infty} \delta^t g_i^t.$$

A strategy is *public* if it depends only on public history. A *public perfect equilibrium (PPE)* is a profile of public strategies that, beginning at any date t and given any public history h^{t-1}, form a Nash equilibrium from that point on.

$E(\delta) \subseteq V^*$: set of discounted average PPE payoff vectors.

Remark: Pure strategy sequential equilibria from APS are PPE in pure strategies.

Big question: Under what conditions is a convex subset W of V^* with a twice continuously differentiable boundary achievable in equilibrium for δ close to 1?

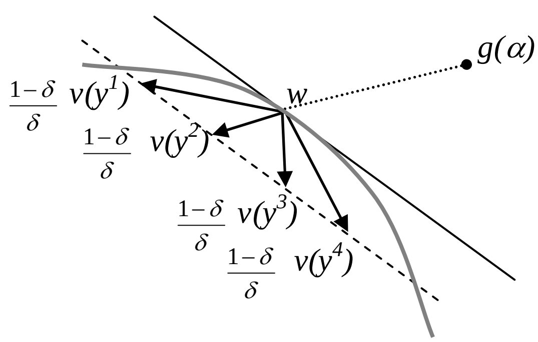


Claim 1. If any w on the boundary of W is generated using continuation values on a tangent hyperplane, then W is achievable in equilibrium for δ sufficiently close to 1.

What does it mean for w to be generated with continuation values on a tangent hyperplane?

It means that there is a mixed action profile α with payoff $g(\alpha)$ separated from V^* by the tangent hyperplane and transitions $v(y)$ for each signal $y \in Y$ such that

- $w = (1-\delta)g(\alpha) + \delta \sum_{y \in Y} \pi(y \mid \alpha)\left(w + \frac{(1-\delta)v(y)}{\delta} \right)$
- players have incentives to follow α given those continuation values:
  $(1-\delta)g_i(\alpha) + \delta \sum_{y \in Y} \pi(y \mid \alpha)\left(w_i + \frac{(1-\delta)v_i(y)}{\delta} \right)$
  is maximized by the actions in the support of α.
- $v(y) \cdot N = const$ for all signals y, where N is the unit normal vector at w



Note that $w + \frac{(1-\delta)v(y)}{\delta}$ is inside W for δ sufficiently close to 1, since the boundary of W is C^2.

Proof. If *w* on the boundary of *W* is generated using continuation values on a tangent hyperplane, then for δ sufficiently close to 1, a neighborhood around *w* is generated using the same mixed action profile and similar transitions. These neighborhoods for all *w* cover the boundary of *W*. Since the boundary is compact, we can pick a finite subcover and a uniform bound $\underline{\delta}$ such that the entire boundary of W is generated by W for $\delta \geq \underline{\delta}$. Then with public randomization the whole set W is generated.

<u>More notation:</u>

$|A_i| = m_i$, $|Y| = m$; $m_i \times m$ matrix whose rows correspond to actions and columns to signals

$$\Pi_i(\alpha_{-i}) = \pi(\cdot \mid \cdot, \alpha_{-i}).$$

<u>Definition</u>. A mixed strategy profile α has *individual full rank* if $\Pi_i(\alpha_{-i})$ has rank m_i for all players. It has *pairwise full rank* if

$$\Pi_{ij}(\alpha) = \begin{pmatrix} \Pi_i(\alpha_{-i}) \\ \Pi_j(\alpha_{-j}) \end{pmatrix}$$

has rank $m_i + m_j - 1$ for all pairs of players.

Intuitively, it means that any deviations of different players can be statistically distinguished, and any two deviations of the same player can be statistically distinguished.

Two types of hyperplanes: coordinate hyperplanes (orthogonal to one of coordinate axes) and regular (all other hyperplanes).

Claim 2. If a profile has individual full rank, then any action profile α in which player *i* chooses a static best response is enforceable on player *i*'s coordinate hyperplane.

Key idea behind the proof. On this hyperplane we must keep player *i*'s continuation value fixed, but we can independently vary all other players' continuation values in any way we want. Due to individual full rank, we can find continuation values that satisfy player *j*'s incentive constraints with equality.

Claim 3. If a profile has pairwise full rank, then it is enforceable on all regular hyperplanes with all incentives constraints satisfied with equality.

Key idea for the proof: The incentive constraints form a system of linear equations (in which payoffs are unknowns), which must have a solution if the rank conditions are satisfied.

Folk Theorem. Suppose all pure action profiles have individual full rank, and there exists a profile α that has pairwise full rank. Then

a. there exists an open dense set of profiles each of which has pairwise full rank
b. the minmax Folk Theorem holds.

Note: FLM also present other versions of the Folk Theorem.

Proof. We will not prove (a) (see FLM). For (b), for any point w on the boundary of W with a regular tangent hyperplane, we can find a profile with pairwise full rank whose payoff is separated from W by that hyperplane. Then by claim 3 w can be generated using that profile and continuation values on a tangent hyperplane. If w has a coordinate tangent hyperplane of player *i* then either w maximizes the payoff of player i or minimizes it. In the former case, it can be generated using the action profile that maximizes the payoff of player *i* (claim 2). In the latter case, it can be generated using the action profile that minmaxes player *i* (also claim 2). QED