Economics 209A
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Solutions to Problem Set 6.

Problem 1.

Consider education signaling game with three types, whose natural productivities are $x_L < x_M < x_H$. Assume that education is unproductive and per-unit cost of getting education is $c_L > c_M > c_H$ for each of the three types. Market pays perceived productivity to each worker type.

(a) Please characterize all triples of education levels (e_L, e_M, e_H) that are part of a separating equilibrium.

There is a separating equilibrium with a triple of education levels (e_L, e_M, e_H) if and only if

$$e_L = 0,\; e_M \in \left[\frac{x_M - x_L}{c_L}, \frac{x_M - x_L}{c_M}\right] \quad and \quad e_H \in \left[e_M + \frac{x_H - x_M}{c_M}, e_M + \frac{x_H - x_M}{c_H}\right].$$

(b) Which of those triples are part of a separating equilibrium that satisfies the intuitive criterion? For each triple, specify off-equilibrium path beliefs that satisfy the intuitive criterion.

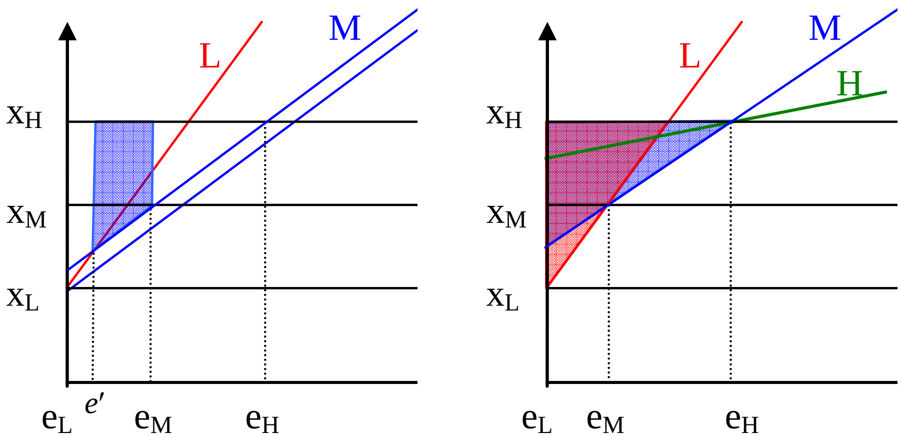
First, for any triple (e_L, e_M, e_H) that satisfies the conditions above, the intuitive criterion implies beliefs that put probability 1 on the high type for all education levels in $\left(e_M + \frac{x_H - x_M}{c_M}, e_H\right]$. *Therefore, we must have* $e_H = e_M + \frac{x_H - x_M}{c_M}$ *for an equilibrium to satisfy the intuitive criterion. Furthermore, the beliefs must put probability 0 on the low type for education levels in* $\left(\frac{x_H - x_L}{c_L}, e_H\right]$ *(since the low type would not want to deviate there even if he is believed to be the high type, but the high type would). Therefore,* $e_M \le \frac{x_H - x_L}{c_L}$ *is a necessary condition for an equilibrium to satisfy the intuitive criterion. For any triple (e_L, e_M, e_H) that satisfies these conditions (summarized below) there are equilibrium beliefs that satisfy the intuitive criterion:*

$$e_L = 0,\; e_M \in \left[\frac{x_M - x_L}{c_L}, \min\left(\frac{x_M - x_L}{c_M}, \frac{x_H - x_L}{c_L}\right)\right] \quad \text{and} \quad e_H = e_M + \frac{x_H - x_M}{c_M}.$$

One set of such beliefs are L on [e_L, e_M), M on [e_M, e_H) and H on [e_H, ∞).

(c) Of the equilibria in part (b), prove that the only one that passes the D1 criterion is the least cost separating equilibrium.

Consider a triple from (b) with $e_M > \frac{x_M - x_L}{c_L}$. *Then for any* $e \in (e', e_M)$ *the range of wage levels for which the low type may want to deviate (i.e. the set* $D_L^0 \cup D_L$*, the area above the red indifference line of type L) is a subset of the range of wage levels for which the medium type would deviate (i.e. the set* D_M*, the blue shaded area). We have* $D_L^0 \cup D_L \subseteq D_M$ *for* $e \in (e', e_M)$*, so the low type can be 'pruned' from the beliefs of the population. But they type M would deviate to a lower education level, so any separating equilibrium with* $e_M > \frac{x_M - x_L}{c_L}$ *fails D1.*



It is easy to see (graphically) that the converse is true: the least cost separating equilibrium passes D1. Beliefs L on $[e_L, e_M)$*, M on* $[e_M, e_H)$ *and H on* e_H*, and arbitrary beliefs above* e_H *satisfy D1.*

<u>Problem 2.</u>

Consider the Prisoners' Dilemma game from Sannikov (2005). Recall that players learn about each other's actions through signals

$$dX_t^1 = A_t^1\,dt + dZ_t^1 \qquad dX_t^2 = A_t^2\,dt + dZ_t^2$$

and expected stage-game payoffs are given by $g_i(a_i, a_j) = 2a_j - a_i$. Prove that $\mathcal{E}(r)=\mathcal{N}$ for $r = 3$.

Hint: Derive an upper bound on the curvature of $\mathcal{E}(r)$ from the optimality equation, and show the curvature of any convex smooth subset of V^* must exceed this upper bound.

<u>Bonus:</u> Using paper-and-pencil calculations, find as small value of r as you can, for which $\mathcal{E}(r)=\mathcal{N}$.

Recall the optimality equation

$$\kappa = \max_{a \in \mathcal{A} \backslash \mathcal{A}^N} \frac{2\,(g(a) - w)\cdot \mathrm{N}}{r\,(\gamma_1(a)^2/t_1^2 + \gamma_2(a)^2/t_2^2)}$$

For any w on the boundary of a symmetric about the 45-degree line bulb-shaped set (a candidate $\mathcal{E}(r)$), we have $(g(a) - w)\cdot N \leq 3/2\sqrt{2}$. Also, for any non-Nash pair of actions a and any unit vector T, $\gamma_1(a)^2/t_1^2 + \gamma_2(a)^2/t_2^2 \geq 1$. Therefore, we have the following upper bound on the curvature of the set $\mathcal{E}(r)$ at all points other than $\mathcal{N}$: $3\sqrt{2}/r$. Suppose r = 3. Then any set with curvature $\leq \sqrt{2}$ must have diameter at least $2\sqrt{2}$. However, the diameter of the set V^* is $3/2\sqrt{2} < 2\sqrt{2}$. Therefore, if the set $\mathcal{E}(r)$ was bigger than $\mathcal{N}$, it would not fit in V^*, a contradiction.

Problem 3.

This problem intends to apply the ideas of Fudenberg, Levine and Maskin (1994) to PPE in pure strategies. Assume for simplicity that there are only two players. First, let us recall the following two definitions from FLM.

Definition 1. A pure-action profile *a* is *enforceable* if there exists a map v : Y → R^n such that

$$g_i(a) + \sum_{y\in Y} \pi(y\,|\,a) v_i(y) \geq g_i(a_i', a_{-i}) + \sum_{y\in Y} \pi(y\,|\,a_i', a_{-i}) v_i(y)$$

for all $i = 1,2$ and $a_i' \neq a_i$.

Definition 2. A pure-action profile *a* is *pairwise identifiable* if the rank of matrix

$$\Pi_{12}(a) = \begin{pmatrix} \Pi_1(a_2) \\ \Pi_2(a_1) \end{pmatrix}$$

equals rank ($\Pi_1(a_2)$) + rank ($\Pi_2(a_1)$) - 1.

Also, recall Lemma 5.5 from FLM:

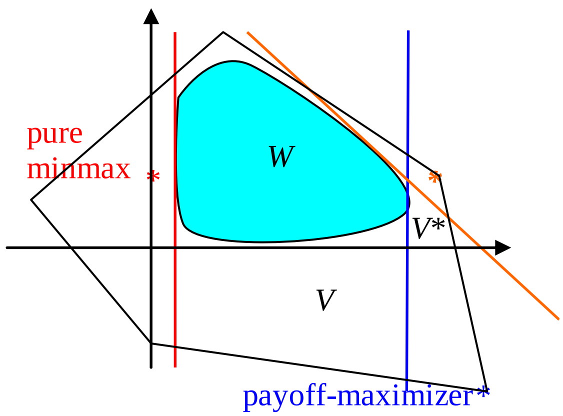
Lemma. If a pure-action profile is enforceable and pairwise-identifiable, then it is enforceable with respect to all regular hyperplanes.

Now, assuming that all pure-strategy profiles in a certain game are enforceable and pairwise-identifiable, sketch a proof the pure minmax-threat Folk Theorem for PPE in pure strategies.

<u>Sketch of Proof</u>. First, the assumption that all pure-action profiles are enforceable and pairwise-identifiable implies (by Lemma 5.5) that they are enforceable on all regular hyperplanes. By Lemma 5.2 of FLM any enforceable pure-action profile with a best response property for one of the players is enforceable on that player's coordinate

hyperplane. Therefore, the profile that maximizes the payoff of player i and the pure-action minmax profile of player i are enforceable on player i's coordinate hyperplane.

Now, to prove the minmax-threat Folk Theorem consider a convex smooth set W in the interior of V^*, and let us show that there is a pure action profile with payoffs separated from W by a tangent hyperplane, which is enforceable on that tangent hyperplane. We have already shown that for regular hyperplanes, any pure action profile is enforceable. For coordinate hyperplanes, either the profile that maximizes the payoff of the appropriate player or the profile that minmaxes him is separated from W by the tangent hyperplane, and is enforceable by the previous paragraph. Thus implies the Folk Theorem. QED



Remark: *This proof of the Folk Theorem does not require individual full rank assumption, which requires many more signals (roughly as many as actions) than pairwise identifiability (which requires only three signals). Enforceablity of pure-action profiles is much broader than individual full rank, and holds in many settings where individual full rank fails.*