Economics 209A
Professor Yuliy Sannikov

Solutions to Problem Set 5.

Problem 1.

There is a seller with valuation 0 for an object and a potential buyer, whose valuation is distributed uniformly on the interval [a, b] with $a > 0$. The buyer's actual valuation is his private information. Both the seller and the buyer discount future payoffs at rate r. There is a natural price path exogenously given by

$$p(t) = a + (b-a)\, e^{-rt}$$

At each moment of time $t \in [0, \infty)$ the seller can charge price $p(t)$ or a, and the buyer must decide whether to buy at the current price or not.

(a) Show that if the seller asks for price a, the all buyer types will immediately buy.

The lowest price the buyer can ever expect is a, which is above each buyer's valuation. Therefore, if the price drops to a, then everybody must buy immediately.

(b) Denote by $T(x) \in [0, \infty]$ the time when the buyer with valuation x decides to buy the item in case the seller does not drop the price to a. Show that $T(x)$ is weakly decreasing in x in any equilibrium.

Suppose not. Then there are x and x′ > x such that T(x) < T(x′). Then the payoff that buyer x gets from conceding at T(x) must be weakly better than her payoff from waiting until T(x′), i.e.

$$\int_0^{T(x)} e^{-rt}(x-a)dF(t) + e^{-rT(x)}(x-b(T(x)))(1-F(T(x)) \geq \int_0^{T(x')} e^{-rt}(x-a)dF(t) + e^{-rT(x')}(x-b(T(x')))(1-F(T(x'))$$

$$\Leftrightarrow e^{-rT(x)}(x-b(T(x)))(1-F(T(x)) \geq \int_{T(x)}^{T(x')} e^{-rt}(x-a)dF(t) + e^{-rT(x')}(x-b(T(x')))(1-F(T(x'))\qquad (1)$$

Similarly, buyer x′ gets a weakly better payoff from waiting until T(x′), i.e.

$$e^{-rT(x)}(x'-b(T(x)))(1-F(T(x)) \leq \int_{T(x)}^{T(x')} e^{-rt}(x'-a)dF(t) + e^{-rT(x')}(x'-b(T(x')))(1-F(T(x'))\qquad (2)$$

Subtracting (2) from (1) we get

$$e^{-rT(x)}(x-x')(1-F(T(x)) \geq \int_{T(x)}^{T(x')} e^{-rt}(x-x')dF(t) + e^{-rT(x')}(x-x')(1-F(T(x'))$$

$$\Leftrightarrow e^{-rT(x)}(1-F(T(x)) < \int_{T(x)}^{T(x')} e^{-rt}dF(t) + e^{-rT(x')}(1-F(T(x'))$$

since (x - x′) < 0. This leads to a contradiction because

$$e^{-rT(x)}(1-F(T(x)) = \int_{T(x)}^{T(x')} e^{-rT(x)}dF(t) + e^{-rT(x')}(1-F(T(x')) > \int_{T(x)}^{T(x')} e^{-rt}dF(t) + e^{-rT(x')}(1-F(T(x'))$$

(c) Denote by F(x) the density of the seller's concessions. Denote by τ the time when F reaches 1. Argue that on (0, τ] function F(t) must be monotonically increasing towards 1, and T(x) has an inverse v(t) which must be monotonically decreasing.

By the standard war of attrition argument T(x) cannot be flat (otherwise the seller will not concede in an interval just before the time when T(x) is flat, and so buyers are better off buying slightly earlier). Therefore, T(x) is monotonically decreasing, so it has an inverse v(t). By the standard war of attrition argument, v(t) and F(t) cannot have atoms on (0, T]. Furthermore, before time T, v(t) or F(t) cannot be flat alone, and v(t) and F(t) cannot be flat at the same time. Therefore, F(t) is monotonically increasing towards 1 on (0, T] and v(t) is monotonically decreasing on (0, T].

(d) Write the seller indifference conditions and show that it implies that τ < ∞. Then, write a first order condition for the optimal concession time of a buyer of type v(t). From this condition, derive a differential equation for F and solve it to find the equilibrium.

Let us derive the equations that v(t) and F(t) must satisfy. Seller's indifference implies that

$$-\frac{1}{b-a}\int_0^t e^{-rs}p(s)v'(s)ds + e^{-rt}a\frac{v(t)-a}{b-a} = const \Rightarrow$$

$$(a-p(t))v'(t) = ra(v(t)-a)$$

$$\frac{v'(t)}{v(t)-a} = \frac{ra}{a-p(t)} \Rightarrow \log(v(t)-a) = \int_0^t \frac{ra}{a-p(s)}ds + K_V\ ,$$

where $K_V = \log(v(t)-a) \le \log(b-a)$ *. Note that* $\int_0^\infty \frac{ra}{a-p(s)}ds = -\infty$ *implies that* $\tau < \infty$*.*

The utility of a seller with valuation w from waiting until time t is

$$\int_0^t e^{-rs}(w-a)f(s)ds + e^{-rt}(w-p(t))(1-F(t))$$

The first-order condition gives

$$e^{-rt}(p(t)-a)f(t) + e^{-rt}p'(t)(1-F(t)) - re^{-rt}(v(t)-p(t))(1-F(t)) = 0$$

$$e^{-rt}(p(t)-a)f(t) - re^{-rt}(p(t)\text{-}a)(1-F(t)) - re^{-rt}(v(t)-p(t))(1-F(t)) = 0$$

$$\frac{f(t)}{(1-F(t))} = \frac{r(v(t)\text{-}a)}{(p(t)-a)} \Rightarrow \log(1-F(s)) = -\int_0^t \frac{r(v(s)\text{-}a)}{(p(s)-a)}ds - K_F$$

It follows that $\tau = \infty$ *or 0. Because we already found earlier that* $\tau < \infty$*, it must be that* $\tau = 0$*, i.e. the seller must concede with probability 1 at time 0.*

Problem 2.

For their computational procedure in Theorem 5, APS 90 assume that $W_0 = W \subseteq R^N$ is compact and $V \subseteq B(W) \subseteq W$. This problem explores the importance of these assumptions.

(a) Under the assumptions of APS, what is $B(R^N)$?

Answer: $B(R^N) = R^N$. *Let us prove this. For any* $w \in R^N$, *we need to show that* $w \in B(R^N)$. *Let* q^N *be a pure strategy Nash equilibrium of the stage game. Let* w' *be defined by*

$$w = (1-\delta)\Pi(q^N) + \delta w'.$$

Then with $u(p) = w'$ *for all signals* p, (q^N, u) *is an admissible pair with respect to* R^N *that generates* w.

(b) Suppose $W_0 = W$ is compact and $V \subseteq W$; but it is not necessarily true that $B(W) \subseteq W$. Is it still true that W_n converges to V as $n \to \infty$? Prove or give a counterexample.

Answer: Yes, it is true. Let $\bar{W}_0$ *be a sufficiently large set that contains* W_0 *and satisfies* $B(\bar{W}_0) \subset \bar{W}_0$. *(Such a set can be found easily. Without loss of generality assume that the set of all feasible payoff pairs* $\bar{V}$ *contains the origin. Then* $B(M\bar{V}) \subset M\bar{V}$ *for all M > 1. We can take* $\bar{W}_0 = M\bar{V}$ *for a sufficiently large M.) Then APS implies that* $B^n(\bar{W}_0)$ *is a decreasing sequence that converges to V. Moreover, by induction on n it follows easily that*

$$V \subseteq W_n \subseteq B^n(\bar{W}_0)$$

because the operator B is monotonic. Therefore, W_n *must converge to V.*

Remark *(From Asaf Plan's solution): We can also let* $\bar{W}_0 = \text{co } W_0$ *and carry out the argument as above.*

(c) Suppose that $W_0 = W \subseteq R^N$ is compact and $W \subseteq B(W)$. Is it true that $\{W_n\}$ is an increasing sequence that converges to V? Prove or give a counterexample.

Answer: Definitely not. Suppose that the stage game has a unique pure strategy Nash equilibrium with payoff $\Pi(q^N)$. *Let* $W_0 = \{\Pi(q^N)\}$. *Then* $B(W_0) = W_0$ *and* W_n *does not converge to* V *(unless the repetition of static Nash is the only sequential equilibrium).*

<u>Problem 3.</u>

Consider the costly state verification setting with a risk-neutral principal and a risk-neutral agent, in which the project's returns are distributed uniformly on the interval $[0, \bar{y}]$ and the verification cost is $c \leq \bar{y}/2$.

(a) Assume that the investor can commit to a contract. What is the maximal amount of capital K that the agent can raise?

We know that the optimal contract for any capital level K is a debt contract. If debt level is D, such a contract raises

$$\frac{D}{\bar{y}}\left(\frac{D}{2} - c\right) + \frac{\bar{y} - D}{\bar{y}} D = \frac{D(\bar{y} - D/2 - c)}{\bar{y}}$$

This expression is maximized when D = $\bar{y}$-c, so the maximal amount of capital that the agent can raise is $\frac{(\bar{y}-c)^2}{2\bar{y}}$.

(b) Now, suppose that the investor cannot commit, and a contract can only give a right, but not an obligation to verify. What is the maximal amount of capital that the agent can raise in this case when $c \le \bar{y}/3$?

The "most credible" contract corresponding to the contract in part (a) is credible when c $\le \bar{y}/3$, *so the maximal amount of capital is still* $\frac{(\bar{y}-c)^2}{4\bar{y}}$.

(c) If the investor cannot commit, what is the maximal amount of capital that the agent can raise in this case when $c \in (\bar{y}/3, \bar{y}/2]$?

Consider any credible contract and denote by D the level such that the agent can get away without monitoring by paying D. Then all agents in the interval [0, D] get monitored. All agents in the interval [D, $\bar{y}$*] pay at most D. Letting* $p \ge D/\bar{y}$ *be the mass of agents who are monitored, we must have*

$$(p - D/\bar{y})D + D^2/(2\bar{y}) \ge cp \Rightarrow p \ge \frac{D^2}{2\bar{y}(D-c)}$$

Therefore, the amount of capital raised by such a contract is at most

$$D(1-D/\bar{y}) + D^2/(2\bar{y}) - cp \le D(1-D/\bar{y}) + D^2/(2\bar{y}) + c\frac{D^2}{2\bar{y}(D-c)},$$

where the bound is reached by a contract, in which the interval of agents who are monitored is $\left[0, \frac{D^2}{2(D-c)}\right]$ *and everybody pays min(y, D). If the investor monitors on a set* $[0, D']$, *then to maximize the expected revenue, it is best to extract everything on the monitoring set and extract D′ otherwise. Therefore, we conclude that the revenue-maximizing credible contract must be a standard debt contract with debt level D that satisfies*

$$D = \frac{D^2}{2(D-c)} \Rightarrow D = 2c$$

This contract raises capital $K = \frac{D(\bar{y} - D/2 - c)}{\bar{y}} = \frac{2c(\bar{y}-2c)}{\bar{y}}$.

<u>Problem 4.</u>

Consider a repeated Prisoners' Dilemma with expected stage-game payoffs given by

	C	D
C	π,π	$-b,\pi+g$
D	$\pi+g,-b$	0,0

Suppose that players do not see each other's actions, but only see a public signal $s = 0, 1$ at the end of each period, whose probability distribution is given as follows, for some $\lambda > \mu > 0$

	(C,C)	(C,D) or (D,C)	(D,D)
Prob(s=1)	λ	μ	0
Prob(s=0)	$1-\lambda$	$1-\mu$	1

Suppose that the actual payoff that each player gets in a stage game depends only on his action and the public signal. Please find the payoff of player i as a function of his action $q_i = C, D$ and signal $s = 0, 1$.

The answer to this problem is found by solving simultaneous equations:

$$\Pi_1(C,C) = \pi = \lambda\pi_1(C,1) + (1-\lambda)\pi_1(C,0)$$
$$\Pi_1(C,D) = -b = \mu\pi_1(C,1) + (1-\mu)\pi_1(C,0)$$
$$\Pi_1(D,C) = \pi + g = \mu\pi_1(D,1) + (1-\mu)\pi_1(D,0)$$
$$\Pi_1(D,D) = 0 = \pi_1(D,0)$$

for $\pi_1(C,1)$, $\pi_1(C,0)$, $\pi_1(D,1)$, $\pi_1(D,0)$, where Π is given by the matrix of expected payoff above. Solving these equations, we find that

$$\pi(D,0) = 0,\ \pi(D,1) = \frac{\pi+g}{\mu},\ \pi(C,0) = \frac{\pi\mu + b\lambda}{\mu-\lambda} \text{ and } \pi(C,1) = \frac{\pi(\mu-1)+b(\lambda-1)}{\mu-\lambda}.$$

Problem 5.

Let $dX_t = \mu\, dt + \sigma\, dZ_t$, where Z is a standard Brownian motion. Find the drift and volatility of $S_t = e^{X_t}$.

Solution (by Hugo Salgado):

This is an application of the Ito's Lemma. Let consider as $S_t = f(X_t) = e^{X_t}$. Then, by Ito's formula we ge that:

$$dS_t = df(X_t) = \left(S_t\mu + \frac{1}{2}S_t\sigma^2\right)dt + S_t\sigma dZ_t$$

Therefore, the drift of S_t is given by $\left(S_t\mu + \frac{1}{2}S_t\sigma^2\right)$ and the volatility of S_t is given by $S_t\sigma$.