Economics 209A
Professor Yuliy Sannikov

Hints to Problem Set 5 (due **Wednesday, November 23**)

Problem 1.

There is a seller with valuation 0 for an object and a potential buyer, whose valuation is distributed uniformly on the interval $[a, b]$ with $a > 0$. The buyer's actual valuation is his private information. Both the seller and the buyer discount future payoffs at rate r. There is a natural price path exogenously given by

$$p(t) = a + (b-a)\, e^{-rt}$$

At each moment of time $t \in [0, \infty)$ the seller can charge price $p(t)$ or a, and the buyer must decide whether to buy at the current price or not.

(a) Show that if the seller asks for price a, the all buyer types will immediately buy.

Note that a is the lowest price that the seller can ever charge.

(b) Denote by $T(x) \in [0, \infty]$ the time when the buyer with valuation x decides to buy the item in case the seller does not drop the price to a. Show that $T(x)$ is weakly decreasing in x in any equilibrium.

Argue by contradiction. Suppose there are x and $x' > x$ such that $T(x) < T(x')$. Write down the condition that the expected payoff buyer x from waiting until time $T(x)$ is weakly better than that from waiting until $T(x')$. Write a similar condition for the other buyer. Show that the two conditions cannot hold simultaneously.

(c) Denote by $F(x)$ the density of the seller's concessions. Denote by τ the time when F reaches 1. Argue that on $(0, \tau]$ function $F(t)$ must be monotonically increasing towards 1, and $T(x)$ has an inverse $v(t)$ which must be monotonically decreasing.

Use the standard war of attrition arguments to rule out the cases where $T(x)$, $F(t)$ and $v(t)$ are flat or have atoms. You do not need to go into too many details on this.

(d) Write the seller indifference conditions and show that it implies that $\tau < \infty$. Then, write a first order condition for the optimal concession time of a buyer of type $v(t)$. From this condition, derive a differential equation for F and solve it to find the equilibrium.

For the seller, you will derive an ODE for $v(t)$ that must hold if the seller is indifferent between all concession times until τ. From this equation, you will find that $v(t)$ would hit a in finite time. Conclude that the seller must stop conceding in finite time. From the first order condition for the optimal concession time of a buyer of type $v(t)$, you will derive an

ODE for F(t) that involves v(t). From this equation, show that F never reaches 1 unless F(0) = 1. Draw your conclusions.

Problem 2.

For their computational procedure in Theorem 5, APS 90 assume that $W_0 = W \subseteq R^N$ is compact and $V \subseteq B(W) \subseteq W$. This problem explores the importance of these assumptions.

(a) Under the assumptions of APS, what is $B(R^N)$?

Think about the set of values of pairs (q, u) admissible with respect to R^N*, where q is the Nash equilibrium quantity pair and u is a constant function.*

(b) Suppose $W_0 = W$ is compact and $V \subseteq W$; but it is not necessarily true that $B(W) \subseteq W$. Is it still true that W_n converges to V as $n \to \infty$? Prove or give a counterexample.

Yes, it is true, and you need to prove it. What if there is a set $\overline{W}_0$ *that contains* W_0*, which satisfies the APS assumptions to start an algorithm? Is there such a set* $\overline{W}_0$ *? You need to have very good APS intuition to figure this out.*

(c) Suppose that $W_0 = W \subseteq R^N$ is compact and $W \subseteq B(W)$. Is it true that $\{W_n\}$ is an increasing sequence that converges to V? Prove or give a counterexample.

Think about the Nash equilibrium.

Problem 3.

Consider the costly state verification setting with a risk-neutral principal and a risk-neutral agent, in which the project's returns are distributed uniformly on the interval $[0, \bar{y}]$ and the verification cost is $c \le \bar{y}/2$.

(a) Assume that the investor can commit to a contract. What is the maximal amount of capital K that the agent can raise?

Write the investor's payoff as a function of D. Maximize.

(b) Now, suppose that the investor cannot commit, and a contract can only give a right, but not an obligation to verify. What is the maximal amount of capital that the agent can raise in this case when $c \le \bar{y}/3$?

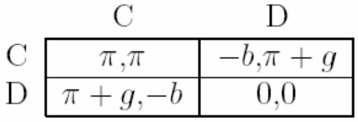
Is the "most credible" contract corresponding to the contract in part (a) credible?

(c) If the investor cannot commit, what is the maximal amount of capital that the agent can raise in this case when $c \in (\bar{y}/3, \bar{y}/2]$?

This is by far the trickiest part of the problem. You will need to identify a credible contract that raises the most capital, which will be less than the amount of capital raised by the contract in part (a). ***You may not rely on the assumption that the credible contract that raises the most capital is a standard debt contract,*** *although you are certainly welcome to derive this conclusion from the first principles. Denote by D the amount such that any agent can get away without monitoring by paying D. Argue that the agents in the interval [0, D] get monitored, and the agents in the interval [D,* $\bar{y}$ *] may or may not get monitored. Find a lower bound on the mass of agents who are monitored for the contract to be credible. From this, write an upper bound on the amount of capital raised by such contract. For what level of D is this upper bound maximized. Can you find a contract that reaches this upper bound?*

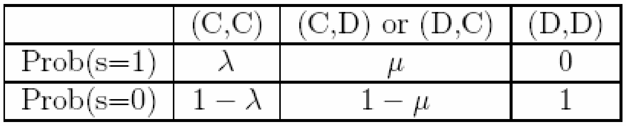
Problem 4.

Consider a repeated Prisoners' Dilemma with expected stage-game payoffs given by



	C	D
C	π,π	$-b,\pi+g$
D	$\pi+g,-b$	0,0

Suppose that players do not see each other's actions, but only see a public signal $s = 0, 1$ at the end of each period, whose probability distribution is given as follows, for some $\lambda > \mu > 0$



	(C,C)	(C,D) or (D,C)	(D,D)
Prob(s=1)	λ	μ	0
Prob(s=0)	$1-\lambda$	$1-\mu$	1

Suppose that the actual payoff that each player gets in a stage game depends only on his action and the public signal. Please find the payoff of player *i* as a function of his action $q_i = C, D$ and signal $s = 0, 1$.

The answer to this problem is found by solving simultaneous equations:

$$\begin{aligned}
\Pi_1(C,C) &= \pi = \lambda\pi_1(C,1) + (1-\lambda)\pi_1(C,0) \\
\Pi_1(C,D) &= -b = \mu\pi_1(C,1) + (1-\mu)\pi_1(C,0) \\
\Pi_1(D,C) &= \pi + g = \mu\pi_1(D,1) + (1-\mu)\pi_1(D,0) \\
\Pi_1(D,D) &= 0 = \pi_1(D,0)
\end{aligned}$$

Problem 5.

Let $dX_t = \mu\, dt + \sigma\, dZ_t$, where Z is a standard Brownian motion. Find the drift and volatility of $S_t = e^{X_t}$.

Use Ito's formula with $f(X) = e^X$.