Economics 209A
Professor Yuliy Sannikov

Problem Set 5.

Directions: Please do any two problems from this problem set and turn them in on Wednesday, November 16, 2005 in my mailbox. On Thursday November 17 I will post hints online for each problem. Please do the rest of the problems by Wednesday November 23. Make every attempt to do each problem on your own. If you need help, please feel free to consult papers or discuss the problems with your classmates. On your solutions, please acknowledge any references you used or help you got, e.g. "I discussed this problem with so and so." In your formal proofs, please be concise.

Problem 1.

There is a seller with valuation 0 for an object and a potential buyer, whose valuation is distributed uniformly on the interval $[a, b]$ with $a > 0$. The buyer's actual valuation is his private information. Both the seller and the buyer discount future payoffs at rate r. There is a natural price path exogenously given by

$$p(t) = a + (b-a)\, e^{-rt}$$

At each moment of time $t \in [0, \infty)$ the seller can charge price $p(t)$ or a, and the buyer must decide whether to buy at the current price or not.

(a) Show that if the seller asks for price a, then all buyer types will immediately buy.

(b) Denote by $T(x) \in [0, \infty]$ the time when the buyer with valuation x decides to buy the item in case the seller does not drop the price to a. Show that $T(x)$ is weakly decreasing in x in any equilibrium.

(c) Denote by F(x) the density of the seller's concessions. Denote by τ the time when F reaches 1. Argue that on $(0, \tau]$ function F(t) must be monotonically increasing towards 1, and $T(x)$ has an inverse $v(t)$, which must be monotonically decreasing.

(d) Write the seller's indifference conditions and show that they imply $\tau < \infty$.

(e) Write a first order condition for the optimal concession time of a buyer of type $v(t)$. From this condition, derive a differential equation for F and solve it to find the equilibrium.

Problem 2.

For their computational procedure in Theorem 5, APS 90 assume that $W_0 = W \subseteq R^N$ is compact and $V \subseteq B(W) \subseteq W$. This problem explores the importance of these assumptions.

(a) Under the assumptions of APS, what is $B(R^N)$?

(b) Suppose $W_0 = W$ is compact and $V \subseteq W$; but it is not necessarily true that $B(W) \subseteq W$. Is it still true that W_n converges to V as $n \to \infty$? Prove or give a counterexample.

(c) Suppose that $W_0 = W \subseteq R^N$ is compact and $W \subseteq B(W)$. Is it true that $\{W_n\}$ is an increasing sequence that converges to V? Prove or give a counterexample.

Problem 3.

Consider the costly state verification setting with a risk-neutral principal and a risk-neutral agent, in which the project's returns are distributed uniformly on the interval $[0, \bar{y}]$ and the verification cost is $c \leq \bar{y}/2$.

(a) Assume that the investor can commit to a contract. What is the maximal amount of capital K that the agent can raise?

(b) Now, suppose that the investor cannot commit, and a contract can only give a right, but not an obligation to verify. What is the maximal amount of capital that the agent can raise in this case when $c \leq \bar{y}/3$?

(c) If the investor cannot commit, what is the maximal amount of capital that the agent can raise in this case when $c \in (\bar{y}/3, \bar{y}/2]$? Is the contract that raises the maximal amount of capital a standard debt contract?

Problem 4.

Consider a repeated Prisoners' Dilemma with expected stage-game payoffs given by

	C	D
C	π,π	$-b,\pi+g$
D	$\pi+g,-b$	0,0

Suppose that players do not see each other's actions, but only see a public signal $s = 0, 1$ at the end of each period, whose probability distribution is given below (where $\lambda > \mu > 0$)

	(C,C)	(C,D) or (D,C)	(D,D)
Prob(s=1)	λ	μ	0
Prob(s=0)	$1-\lambda$	$1-\mu$	1

Suppose that the actual payoff that each player gets in a stage game depends only on his action and the public signal. Please find the payoff of player i as a function of his action q_i = D, C and signal $s = 0, 1$.

Problem 5.

Let $dX_t = \mu\, dt + \sigma\, dZ_t$, where Z is a standard Brownian motion. Find the drift and volatility of $S_t = e^{X_t}$.