Economics 209A
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Solutions to Problem Set 4.

Problem 1.

It is argued in mergers and acquisitions that a raider may benefit from acquiring a toehold before making a takeover bid. This problem is designed to investigate this claim.

There are two bidders in a potential takeover with valuations for the target v_1 and v_2 distributed independently uniformly on [0, 1]. Suppose that the target's value is 0 in case a take-over fails. Bidders 1 and 2 own fractions α_1 and $\alpha_2 \in [0, ½)$ of the target's shares before the auction. Assume that bidder 1 has a bigger toehold, i.e. $\alpha_1 > \alpha_2$. The target owns the remaining fraction $1 - \alpha_1 - \alpha_2 \in [½, 1]$ of its shares. Each bidder needs to acquire ½ of the target's shares to gain control over the target. If bidder *i* gains control over the target, the target's value rises to v_i.

Suppose that the target designs an optimal auction to determine its acquirer. Assume that the target can commit to such an auction. If bidder *i* wins the auction, the target needs to give it $½ - \alpha_i$ of its shares.

(a) If bidder 1 wins the auction and pays x to the acquirer, what is his payoff?

$$½ \, v_1 - x$$

(b) If bidder 2 wins the auction and bidder 1 pays x to the acquirer, what is bidder 1's payoff?

$$\alpha_1 v_2 - x$$

(c) Following Myerson (1981), please design an optimal auction and derive bidder 1's and bidder 2's expected utilities as functions of their valuations. Which bidder gets a higher payoff? Interpret.

Remark. Assume that if one of the bidders refuses to participate in the auction, then the target does not hold auction altogether. For example, if bidder 1 refuses to participate, then bidder 2 does not acquire the target and the toehold of bidder 1 has value 0. Also, assume that the target knows the toeholds of each bidder because by law, potential acquirers are required to report their toeholds.

The total surplus (expected sum of utilities of the target and each bidder) is given by

$$\int_0^1 \int_0^1 \left(v_1 p_1(v_1, v_2) + v_2 p_2(v_1, v_2) \right) f(v) dv_1 dv_2$$

Bidder i's valuation of ½ of the target's shares is $v_i/2$. *Similarly as in Myerson, it is true that if bidder i with valuation* v_i *pretends to have valuation* w_i, *then his utility is given by*

$$U_i(p,x,w_i)+\frac{v_i-w_i}{2}Q_i(p,w_i).$$

Indeed, he gets the same utility as type w_i *if he loses the auction (in particular, if* $i = 1$ *then his utility from the appreciation of a fraction* α *of the target's shares is the same), but if he wins the auction, his value ½ of the target's shares is* $v_i/2$, *not* $w_i/2$.

Therefore, $\frac{\partial}{\partial v_i}U_i(p,x,v_i)=\frac{1}{2}Q(p,v_i)$ *and* $U_i(p,x,v_i)=U_i(p,x,0)+\int_0^{v_i}\frac{1}{2}Q(p,w_i)dw_i$.

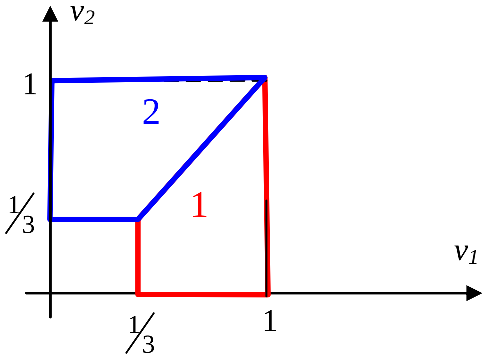
The expected utility of bidder i in an auction defined by (p, x) is

$$U_i(0)+\int_0^1\int_0^1(1-F_i(v_i))\tfrac{1}{2}\,p_i(v_1,v_2)\,f_{-i}(v_{-i})dv_1dv_2$$

and the target's expected payoff is given by

$$\int_0^1\int_0^1\sum_{i=1,2}\left(v_ip_i(v_1,v_2)-\frac{1-F_i(v_i)}{2f_i(v_i)}\right)f(v)dv_1dv_2-U_1(0)-U_2(0)$$

Because $-\frac{1-F_i(v_i)}{2f_i(v_i)}=\tfrac{1}{2}(v_i-1)$, *the allocation in the optimal auction is determined by the comparison of* $\tfrac{3}{2}v_1-\tfrac{1}{2}$, $\tfrac{3}{2}v_2-\tfrac{1}{2}$ *and 0. Graphically, the allocation is determined as follows:*



One way to implement this auction is to require each bidder to sell its toehold to the target at price 0 as a participation requirement, and then run a second-price auction with reservation price 1/3.

The expected utility of bidder i with valuation $v_i \geq \frac{1}{3}$ *is*

$$U_i(v_i)=\int_{\frac{1}{3}}^{v_i}w_idw_i\ ,$$

which does not depend on the bidder's initial toehold.

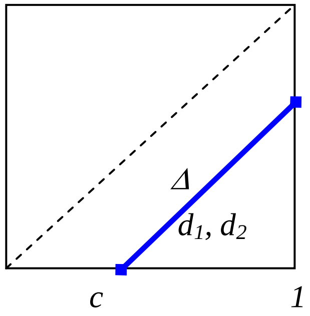
Although this problem has many questionable assumptions, especially the target's ability to commit in an auction, it illustrates the following intuition: when a potential raider *acquires a toehold, the target's management may have ways to exploit it and extract more surplus from the raider. For a detailed analysis of these issues, including empirical facts, see Betton, Eckbo and Thorburn "The Zero-Toehold Puzzle," working paper, Dartmouth College.*

Problem 2.

Consider a Rubinstein bargaining setting, in which both players have the same constant cost c of waiting per period.

(a) Find A and B.

Rubinstein (1982) implies that A = B = [c, 1], as illustrated in the following figure:



(b) Find the range of values of c, for which there exists a SPE in which agreement is not reached in the first period. Construct such a SPE. As building blocks, you may use SPE with immediate agreement and any value in Δ for the player who makes an offer without elaborating on what happens in those SPE.

There is a SPE with disagreement in the first period if and only if $c \in [0, 1/3]$.

Let us show that if $c > 1/3$, *then agreement must be reached in the first period. Suppose not: player 1 makes the first offer, which is rejected. Then in period 2 player 2 must get a payoff of at least c and player 1 must get a payoff at most 1-c, which is worth at most 1 – 2c in period 1. Therefore, player 1's payoff in a SPE in which the first offer is rejected is at most* $1\text{-}2c < c$, *a contradiction.*

Suppose that $c \leq 1/3$, *and let us construct an SPE with disagreement in the first period. Suppose that player 1's first offer is 1, which is rejected. If the first offer is 1, then starting in period 2, let players play an SPE with immediate agreement and value c to player 2 and 1-c to player 1. If the first offer was in the interval [c, 1), let it be rejected and followed by an SPE with value 1 to player 2 and 0 to player 1. If the first offer was in the interval [0, c), let it be accepted, and if rejected, followed by an SPE with value c to player 2 and 1-c to player 1.*

Problem 3.

Consider the following general set of mechanisms (M, *a*) in the costly state verification framework. The entrepreneur has to choose some message from a message space M, which determines action according to a rule $a : M \rightarrow A$. Actions can consist of a transfer payment (where the entrepreneur can only choose messages that yield feasible transfer payments), and a deterministic decision rule by the lender on whether to verify. If the lender does verify, the action can include a feasible payment that depends on the type verified. In this setting with perfect commitment, we can invoke the revelation principle (as we did in class) and characterize an optimal mechanism. There are, of course, many other mechanisms (that need not be truth-telling direct revelation mechanisms) that implement the same outcome as our standard debt contract. Let this set of optimal mechanisms be called O.

Now suppose that the lender cannot commit through a mechanism to verify; but rather, the mechanism only gives the lender the right to verify, which the lender may choose not to exercise. The lender will only exercise this right when it is in her interest to do so, given equilibrium actions. We call a mechanism "credible" if the lender chooses to verify whenever he has the right to do so. Whether or not a mechanism in O is credible will of course depend on the exogenously given prior distribution F of cash outcomes, the cost of verification, and D (which depends on the exogenous variable K).

A. Identify the "most credible" mechanism in O. In particular, characterize a mechanism in O that is credible for any set of exogenous parameters (F, c, D) which admits any credible mechanism in O.

The most credible mechanism consists of a message space {0, D} and an action a(0) = {verify, extract everything} and a(D) = {receive D from the agent}.

Let us show that if any other mechanism is credible, than this one is. Consider another credible mechanism (M, a) with messages in $M' \subset M$ *inducing verification. Then*

$$E[y \mid m] \geq c$$

for all messages $m \in M'$*, i.e. the expected amount that the lender extracts for each message in* M' *is bigger than or equal to the verification cost. Using the law of iterated expectations*

$$E[y \mid y \in [0, D]] = E[E[y|m]|M'] \geq c$$

so if any other mechanism in O is credible, then the "most credible mechanism" above is credible.

B. Under what conditions on (F, c, D) will this "most credible" mechanism in fact be credible?

This mechanism will be credible if $\int_0^D y\,dF(y) \geq cF(D).$

Problem 4.

Consider the war of attrition from the first lecture and from problem 1 on problem set 1, except that now both players may be behavioral with probabilities p_1 for player 1 and p_2 for player 2. Recall that two players initially make demands (2,1) and (1,2) about how to split \$3, and wait until one of the players concedes. Both players discount future payoffs at the common rate *r*. The behavioral type is not able to concede. The purpose of this problem is to characterize all mixed strategy Bayesian Nash equilibria. Denote by F_1 the CDF of concession times of the *normal* type of player 1, and by F_2 the CDF of concession times of the *normal* type of player 2.

A. Sketch a proof for each of the following claims. Be as concise as possible.

(1) F_1 and F_2 reach 1 at the same time at some time $T \geq 0$.

Suppose F_1 reaches 1 at time T_1 before F_2. Then player 2 will be convinced that player 1 will never concede after time T_1, so it is optimal for him to concede immediately, a contradiction.

(2) F_1 and F_2 have no atoms, except possibly at 0.

If F_1 had an atom at t, then player 2 would not concede in an interval before t. But then for player 1 it is optimal to concede a bit before t rather than at t.

(3) F_1 and F_2 are strictly increasing on [0, T].

First, it is impossible for F_1 to be flat while F_2 is increasing, because player 2 must not concede in the middle of the interval where F_1 is flat. If F_1 and F_2 are both flat on (t_1, t_2) and increasing after t_2, then one can show that it is better to concede at time $t_1 + \varepsilon$ rather than anywhere on $(t_2, t_2+\varepsilon)$ for both players, which contradicts the fact that F_1 and F_2 are increasing after t_2.

B. Find all Bayesian Nash equilibria, and justify your logic.

Because players 1 and 2 must be indifferent between all concession times on [0, T], we must satisfy

$$F_1(t) = \frac{1}{1-p_1} - k_1 e^{-rt} \qquad \textit{and} \qquad F_2(t) = \frac{1}{1-p_2} - k_2 e^{-rt}$$

with boundary conditions $F_1(T) = F_2(T) = 1$ and either $F_1(0)=0$ or $F_2(0) = 0$. Without loss of generality, let us focus on the case when $p_1 > p_2$. Then player 1 is "strong," so $F_1(0) = 0$,

$$F_1(t) = \frac{1}{1-p_1} - \frac{1}{1-p_1} e^{-rt} \Rightarrow\ e^{-rT} = p_1 \Rightarrow\ 1 = F_2(T) = \frac{1}{1-p_2} - k_2 p_1 \Rightarrow k_2 = \frac{p_2}{p_1(1-p_2)}$$

These functions define the unique BNE. In this BNE player 2 concedes at time 0 with positive probability of $\frac{p_1 - p_2}{p_1(1-p_2)}$.

Problem 5.

Consider the following version of the education signaling model. There is a continuum of types of workers with skill levels t ∈ [0, 2]. Skill level is unobservable, but employers can see worker's education e, and infer from it the worker's type. The cost of getting education e to type t is $(t - e)^2$, where t is the "bliss" education level. If the market believes that a worker is of type t, it will pay the worker wage 2t. Find the fully separating signaling equilibrium.

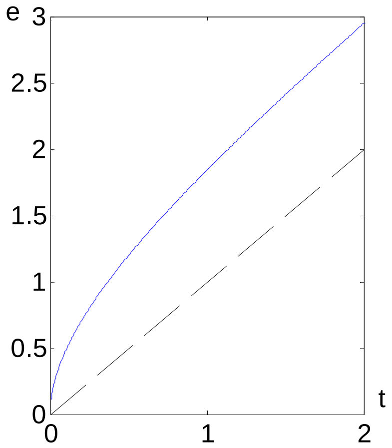
Denote by T(e) the market inference. Then the maximization problem of type t is written as

$$\max_e 2T(e) - (t-e)^2$$

The first order condition gives $2T'(e) = 2(e-T(e))$, *so*

$$T(e) = e - 1 + k\exp(-e)$$

The boundary condition that the worst type takes his preferred education level pins down the constant of integration: $T(0) = -1 + k = 0$, *so* $k = 1$. *We conclude that market inference is given by* $T(e) = e - 1 + \exp(-e)$



and the signaling function is given by the inverse of T. See Figure.

Problem 6.

Recall the setting of Morris and Shin (1998). The state of fundamentals θ is uniformly distributed on the interval [0, 1]. Suppose that the currency is initially pegged at $e^* = 2$, and the exchange rate in the absence of government intervention is given by $f(\theta) = ½+\theta$. There is a unit mass of speculators, each of whom gets a signal x about the state of fundamentals θ uniformly distributed on $[\theta - \varepsilon, \theta + \varepsilon]$ where ε is relatively small. The signals are independent across speculators, conditional on θ. After receiving their signals, the speculators simultaneously decide whether to attack the currency or not. A speculator's payoff is $e^* - f(\theta) - t$ he attacks and the currency is devalued, and $-t$ if attacks and the currency is defended, where $t = 1$ is the transaction cost. If a speculator does not attack, he gets a payoff of 0.

After seeing the mass of speculators who attack, the government decides whether to defend the exchange rate or not. The government derives value $v = 1$ from defending the exchange rate, but has to pay a cost of $c(\alpha, \theta) = 1.2 + \alpha - \theta$, where α is the mass of speculators who attack.

The objective of this problem is to characterize the unique equilibrium.

(a) Find the state of fundamentals $\underline{\theta}$, such that for $\theta < \underline{\theta}$ the government will not defend the currency even if nobody attacks.

The government will not defend the currency even if nobody attacks if $c(0, \theta) > v \Leftrightarrow \theta < 0.2$. *We conclude that* $\underline{\theta} = 0.2$.

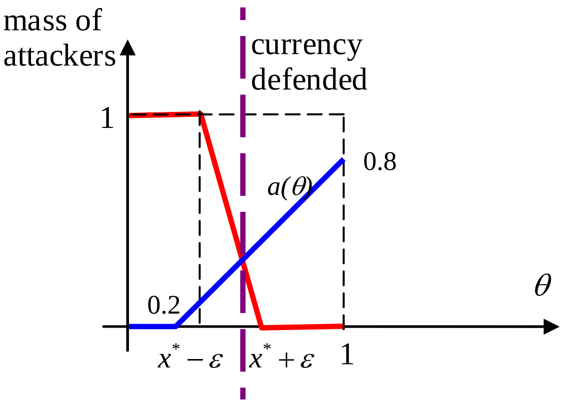
(b) Find the state of fundamentals $\bar{\theta}$, such that for $\theta > \bar{\theta}$ a speculator would get a negative payoff from attacking even if the currency is devalued for sure.

$\bar{\theta}$ *is characterized by* $e^* - f(\bar{\theta}) - t < 0$, *so* $\bar{\theta} = 0.5$.

(c) Find a(θ), the critical mass of speculators that triggers a devaluation when the state of fundamentals is θ.

We have $a(\theta) = 0$ *for* $\theta < \underline{\theta}$ *and* $c(a(\theta), \theta) = v \Rightarrow a(\theta) = \theta - 0.2$ *for* $\theta \geq \underline{\theta}$.

(d) Conjecture that a speculator attacks if and only if his signal $x \leq x^*$. Assuming $x^* \in [\underline{\theta} + \varepsilon, 1 - \varepsilon]$, plot the mass of speculators who attack as a function of the state of fundamentals.



(e) Add *a(θ)* to your plot. Show graphically the region where the currency is defended. As a function of x^*, compute θ^*, the critical value of fundamentals such that the government defends the currency if and only if $\theta > \theta^*$.

See Figure above. Letting $a^*=a(\theta^*)$*, we have* $x^* + \varepsilon - 2\varepsilon a^* = 0.2 + a^* \Rightarrow$

$$a^* = \frac{x^*+\varepsilon-0.2}{1+2\varepsilon} \Rightarrow \theta^* = 0.2 + a^* = \frac{x^*+1.4\varepsilon}{1+2\varepsilon} \Rightarrow x^* = (1+2\varepsilon)\theta^* - 1.4\varepsilon$$

(f) Find θ^* and x^* in equilibrium. You will have a quadratic equation for θ^*.

If a speculator gets signal $x \in [\theta^*-\varepsilon, \theta^*+\varepsilon]$*, he believes that the fundamentals are uniformly distributed on the interval* $[x - \varepsilon, x+\varepsilon]$*. If he attacks the currency, his expected payoff is given by*

$$\frac{1}{2\varepsilon}\int_{x-\varepsilon}^{\theta^*}(e^* - f(\theta))d\theta - t = \frac{1}{2\varepsilon}\int_{x-\varepsilon}^{\theta^*}(1.5-\theta)d\theta - 1 =$$

$$\frac{1}{2\varepsilon}(1.5(\theta^* - x + \varepsilon) - 0.5(\theta^*)^2 + 0.5(x-\varepsilon)^2) - 1$$

Using $x = x^* = (1+2\varepsilon)\theta^* - 1.4\varepsilon$ *in the expression above, this expected payoff must be 0, which implies that*

$$(1+\varepsilon)(\theta^*)^2 - (2.7+2.4\varepsilon)\theta^* + 0.8 + 1.44\varepsilon = 0$$

$$\theta^* = \frac{2.7+2.4\varepsilon - \sqrt{(2.7+2.4\varepsilon)^2 - 4(1+\varepsilon)(0.8+1.44\varepsilon)}}{2(1+\varepsilon)}$$

(g) Is Theorem 2 from Morris and Shin (1998) correct?

No. Letting $\varepsilon \to 0$*, we find that in the limit*

$$\theta^* = \frac{2.7-\sqrt{2.7^2-3.2}}{2} = 0.338812579$$

We have $f(\theta^*)=0.5 + \theta^* = 0.8388 \neq e^* - 2t = 0$*. The intuition that follows Theorem 2 is incorrect, because the speculator who observes* $x = \theta^*$ *is not "marginal" in the sense that he is not indifferent between attacking or not.*