Economics 209A
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Solutions to Problem Set 3

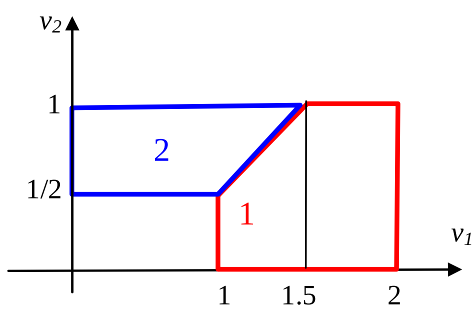
Comments: This problem set was graded on a scale of 10, with possible 2 bonus points (each problem worth 2 points). Overall, the answers were excellent. Problem 4 was by far the hardest, understandably so. Only one person got problem 4 completely right. In Problem 1, a few people found the optimal allocation, but did not suggest any payments. One always needs to be careful to give a full answer. Also, I did not formulate 6c very well, so I did not count it. See the intended interpretation.

Auctions and the revelation principle

Problem 1.

A seller auctions an object to two buyers. The seller's own valuation is 0. The buyers' valuations are distributed uniformly on the intervals [0, 2] and [0, 1] respectively. Find the optimal auction. Illustrates who gets the object for each pair of valuations graphically.

The seller can offer a direct revelation mechanism, in which he compares his own valuation and the virtual valuations of each buyer: $v_1 - \frac{1-F_1(v_1)}{f_1(t_1)} = 2v_1 - 2$ *and* $v_2 - \frac{1-F_2(v_2)}{f_2(t_2)} = 2v_2 - 1$. *The allocation rule can be illustrated as follows:*



The payments are given by

$$x_1(v_1,v_2) = \begin{cases} 0 & \text{if loses} \\ 1 & \text{if wins and } v_2 \le \frac{1}{2} \\ v_2 + \frac{1}{2} & \text{if wins and } v_2 > \frac{1}{2} \end{cases} \qquad x_2(v_1,v_2) = \begin{cases} 0 & \text{if loses} \\ \frac{1}{2} & \text{if wins and } v_1 \le 1 \\ v_1 - \frac{1}{2} & \text{if wins and } v_1 > 1 \end{cases}$$

Problem 2.

This is a hard problem. Consider a setting with a seller and a buyer, whose valuations v_1 and v_2 are drawn independently from the uniform distribution on [0, 1]. Given the seller's valuation, the optimal mechanism to sell the object is to post the price of p = $(1+v_1)/2$ and to sell the object only if the buyer is willing to pay that price. However, what if the seller has an opportunity to design a mechanism before he learns his valuation? Prove that the mechanism above is still optimal.

Let us characterize optimal mechanisms and show that the suggested mechanism delivers the same allocation as the optimal mechanism. We can restrict attention to direct revelation mechanisms, in which the seller and the buyer simultaneously announce their types to an intermediary, and the intermediary determines allocations and payments as functions of the two announcements. In such a mechanism, the total surplus is given by

$$\int_T \left(v_1 p_1(v_1, v_2) + v_2 p_2(v_1, v_2)\right) f(t)dt$$

and the buyer's expected utility is $U_2(0) + \int_T (1 - F_2(v_2)) p_2(v_1, v_2) f_1(v_1) dv$.

The seller's utility is then

$$\int_T (v_1 p_1(v_1, v_2) + \left(v_2 - \frac{1 - F_2(v_2)}{f_2(t_2)}\right) p_2(v_1, v_2)) f(t)dt - U_2(0),$$

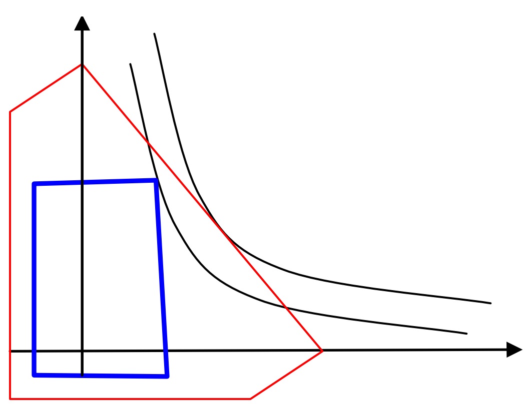
where $v_2 - \frac{1 - F_2(v_2)}{f_2(t_2)} = 2v_2 - 1$. *We find that a mechanism is optimal if and only if* $U_2(0)$ *=0, the seller keeps the item in the event* $v_1 > 2v_2 - 1$ *and gives it to the buyer otherwise (any mechanism that satisfies these properties gives the same payoff to the seller). The mechanism suggested in the problem satisfies these properties.*

<u>Nash bargaining.</u>

<u>Problem 3.</u>

Let c(S) be the Nash bargaining solution relative to the disagreement point (0, 0). Prove or find a counterexample to the following statement: if S is a subset of S′, then c(S′) is at least as good as c(S) to both players. If you find a counterexample, please illustrate it graphically. If you find a proof, please be concise.

Counterexample:



Problem 4.

This problem is based on Nash (1953). Consider a bargaining situation is define by a convex set B, a set of threats for each player A_1 and A_2 and a mapping $u : A_1 \times A_2 \to B$. The bargaining outcome is determined by a game, in which the players choose threats (mixed actions) $t_1 \in \Delta(A_1)$ and $t_2 \in \Delta(A_2)$, and the outcome from B is determined by the Nash bargaining solution relative to the disagreement point $u(t_1, t_2) = (u_1(t_1, t_2), u_2(t_1, t_2))$. Specifically, the payoffs from threats t_1 and t_2 are given by the point $(v_1, v_2) \in B$, which maximizes $(v_1 - u_1(t_1, t_2)) (v_2 - u_2(t_1, t_2))$. The Nash bargaining solution with threats is determined by a mixed strategy Nash equilibrium of this game.

Show that the Nash bargaining solution with threats satisfies the following properties.

(a) The solution (v_1, v_2) is Pareto efficient.

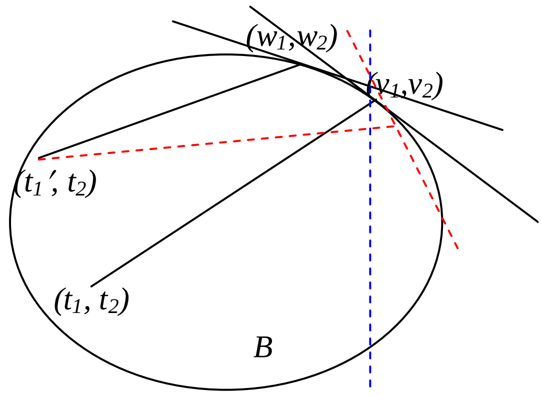
This is true because the Nash bargaining solution relative to any threat point is always Pareto efficient.

(b) For each bargaining game (B, A_1, A_2, u_1, u_2) the solution $(v_1, v_2) \in B$ is unique.

Suppose there were two different solutions (v_1, v_2) and (v_1', v_2') corresponding to threats (t_1, t_2) and (t_1', t_2'). Without loss of generality consider the case when $v_2 < v_2'$. Let (w_1, w_2) be the Nash bargaining solution relative to the threat point given by (t_1, t_2'). Then $v_2 \geq w_2$ because at (t_1, t_2) player 2 does not want to deviate to t_2'. Also $w_1 \leq v_1'$ (and so $w_2 \geq v_2'$) because at (t_1', t_2') player 1 does not want to deviate to t_1. Therefore, $v_2 \geq v_2'$, a contradiction. Intuitively, the statement is true because, effectively, the game has a zero-sum structure.

(c) Consider two bargaining games (B, A_1, A_2, u_1, u_2) and (B', A_1, A_2, u_1, u_2). If (v_1, v_2), the solution of the first game, is an element of B' and $B' \subset B$, then (v_1, v_2) is also chosen from B'.

Let (t_1, t_2) be the threats that correspond to (v_1, v_2). Let us show that these threats are a Nash equilibrium of the bargaining game (B', A_1, A_2, u_1, u_2) also. Take an arbitrary deviation t_1' of player 1. Let (w_1, w_2) and (w_1', w_2') be Nash bargaining solutions from B and B' respectively relative to the threats (t_1', t_2). We would like to show that t_1' is not a profitable deviation in (B', A_1, A_2, u_1, u_2), i.e. $w_1' \leq v_1$. We know that $w_1 \leq v_1$. It can be seen that $w_1' \leq v_1$ graphically. The line connecting the threat point with the solution has the same but opposite slope as the tangent at the solution to the bargaining set. If the bargaining set shrinks, then the solution that corresponds to (t_1', t_2) cannot be to the right of the blue line; otherwise the slope of the line connecting (t_1', t_2) with the solution would be flatter than the tangent slope (look at the red dashed lines). Therefore, player 1 must get at most w_1 from the solution relative to the threat point (t_1', t_2).



(d) A restriction of the set of strategies available to a player cannot increase the value to him of the game.

Suppose that when we restrict the strategies of player 2, the solution moves from (v_1, v_2) *to* (v_1', v_2') *with corresponding to threats* (t_1, t_2) *and* (t_1', t_2')*. Let* (w_1, w_2) *be the Nash bargaining solution relative to the threat point given by* (t_1, t_2')*. Then* $v_2 \geq w_2$ *because at* (t_1, t_2) *player 2 does not want to deviate to* t_2'*. Also* $w_1 \leq v_1'$ *(and so* $w_2 \geq v_2'$*) because at* (t_1', t_2') *player 1 does not want to deviate to* t_1 *(since we did not restrict the strategies of player 1). Therefore,* $v_2 \geq v_2'$*, so player 2 can only get worse off as a result of the restriction.*

<u>Corporate Finance.</u>

<u>Problem 5.</u>

Consider the setting of Bolton and Scharfstein (1990), except with a modification that the entrepreneur has all the bargaining power and the investors act competitively. As before, in each of two periods, the firm needs F of outside funding to operate. If it operates, it gets cash flows π_1 or π_2 with probabilities θ and $1 - \theta$. Cash flows are iid and $\pi_1 < F < \pi_2$. The manager can conceal and divert cash flows, but the investors can force the manager to pay at least π_1. The residual $\pi_2 - \pi_1$ is the non-verifiable component, which the manager can give to the investors only if he has contractual incentives to do so. With competitive investors, the problem is to maximize the manager's expected payoff, subject to the constraint that the investors at least break even:

$$\max_{R_i,\beta_i,R^i} \theta[\pi_1 - R_1 + \beta_1(\bar{\pi} - R^1)] + (1-\theta)[\pi_2 - R_2 + \beta_2(\bar{\pi} - R^2)]$$

s.t. $\quad -F + \theta[R_1 + \beta_1(R^1 - F)] + (1-\theta)[R_2 + \beta_2(R^2 - F)] \geq 0 \qquad$ (investor breaks even)

$\quad \pi_2 - R_2 + \beta_2(\bar{\pi} - R^2) \geq \pi_2 - R_1 + \beta_1(\bar{\pi} - R^1), \qquad$ (truth-telling)

$\quad \pi_i \geq R_i,\ \pi_1 \geq R^i.$

(a) Prove that there exists an optimal contract with $R^1 = R^2 = \pi_1$?

For any contract with $R^i < \pi_1$, we can get a payoff-equivalent contract by increasing R^i to π_1 and decrease R_i by $\beta_i (\pi_1 - R^i)$. With this change all the constraints are still satisfied.

(b) Show that any contract that gives positive profit to the investor can be modified to transfer the investor's profit to the manager.

If not, then we can reduce R_1 and R_2 slightly by the same amount without violating truth-telling. As a result, the agent's payoff will increase at the investor's cost.

(c) Is it true that $\beta_2 = 1$ in the optimal contract?

If we had $\beta_2 < 1$, let us increase β_2 to 1 and decrease R_2 appropriately to maintain $R_2 + \beta_2(R^2 - F) = R_2 + \beta_2(\pi_1 - F)$ unchanged. By this change, more total surplus is generated, but the investor's profit remains unchanged. Therefore, the manager is better off. The truth-telling constraint still holds because this change increases the left hand side of (truth-telling.)

(d) Show that $R_1 = \pi_1$ in the optimal contract?

If not, then we can increase R_1 and β_1 while keeping $-R_1 + \beta_1(\bar{\pi} - \pi_1)$ constant. This adjustment maintains the incentive constraint and increases total surplus, which goes to the investor. By (b), we can transfer the investor's profit to the agent. Therefore, any contract with $R_1 < \pi_1$ is suboptimal since we can modify it to improve the agent's payoff.

(e) Show that the truth-telling constraint is binding.

If it is not binding, we can increase R_2 and β_1 while keeping $\theta\beta_1(\pi_1 - F) + (1-\theta)R_2$ constant. As a result, the investor's profit remains unchanged and the agent's payoff increases.

(f) What is the optimal contract?

By truth-telling $(1-\beta_1)(\bar{\pi} - \pi_1) + \pi_1 = R_2$. Using this in the break-even constraint, we get

$$1 - \beta_1 = \frac{2(F - \pi_1)}{(1-\theta)\bar{\pi} + \theta F - \pi_1} \Rightarrow \beta_1 = \frac{(1-\theta)\bar{\pi} + \pi_1 - (2-\theta)F}{(1-\theta)\bar{\pi} + \theta F - \pi_1}$$

$$R_2 = \frac{2(F - \pi_1)(\bar{\pi} - \pi_1)}{(1-\theta)\bar{\pi} + \theta F - \pi_1} + \pi_1$$

(g) Under what conditions does a feasible contract exist? Compare this condition to one in the case when the investor is a monopolist. Interpret.

The optimal contract exists if the parameters defined by (f) are meaningful. Ultimately, we need $\beta_1 \geq 0$, which is equivalent to

$$F \le \bar{\pi} - (\bar{\pi} - \pi_1)/(2-\theta)$$

This is the same condition as if the investors are competitive. When $F = \bar{\pi} - (\bar{\pi} - \pi_1)/(2-\theta)$, *the optimal contract gives payoffs zero to both the investor and the agent.*

Problem 6.

Again, consider the setting of Bolton and Scharfstein (1990) with competitive investors, except assume this time that the agent also initially has cash $Y \in [0, 2(F - \pi_1)]$, which he can contribute to the project at time 0.

(a) Find R_1, R_2, R^1, R^2, β_1 and β_2 in the optimal contract. (Assume that the investors may be given the ability to disallow the agent to run the project even if he has enough cash to invest).

The contract can be found by the same logic as in the previous problem. We find that $R_1 = R^1 = R^2 = \pi_1$, $\beta_2 = 1$ *and* R_2 *together with* β_1 *are found from*

$R_2 = (1-\beta_1)(\bar{\pi} - \pi_1) + \pi_1$ and

$Y - F + \theta[\pi_1 + \beta_1(\pi_1 - F)] + (1-\theta)[R_2 + \beta_2(\pi_1 - F)] = 0$

We get $1 - \beta_1 = \dfrac{2(F-\pi_1) - Y}{(1-\theta)\bar{\pi} - \pi_1 + \theta F} \Rightarrow \beta_1 = \dfrac{(1-\theta)\bar{\pi} + \pi_1 - (2-\theta)F + Y}{(1-\theta)\bar{\pi} - \pi_1 + \theta F}$

and $R_2 = (1-\beta_1)(\bar{\pi} - \pi_1) + \pi_1$.

(b) Under what conditions does a feasible contract exist?

$$F \le \bar{\pi} - (\bar{\pi} - \pi_1 - Y)/(2-\theta)$$

(c) Find the manager's expected payoff as a function of Y. Compare the derivative of his payoff with respect to Y with 1. Interpret.

The manager's payoff is

$$Y + 2(\bar{\pi} - F) - \frac{\theta(\bar{\pi} - F)(2(F-\pi_1) - Y)}{(1-\theta)\bar{\pi} + \theta F - \pi_1}$$

The derivative of the payoff with respect to Y is $1 + \dfrac{\theta(\bar{\pi} - F)}{(1-\theta)\bar{\pi} + \theta F - \pi_1} > 1$. *The reason for this is that each dollar contributed by the manager improves the efficiency of running the project by reducing the agency problem.*