Economics 209A
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Problem Set 3.

Directions: This problem set is due on Wednesday, October 12, 2005 in my mailbox. You get a little bit more time because it is longer than usual. Make every attempt to do each problem on your own. If you need help, please feel free to consult papers or discuss the problems with your classmates. On your solutions, please acknowledge any references you used or help you got, e.g. "I discussed this problem with so and so." In your formal proofs, please be concise.

Auctions and the revelation principle

Problem 1.

A seller auctions an object to two buyers. The seller's own valuation is 0. The buyers' valuations are distributed uniformly on the intervals [0, 2] and [0, 1] respectively. Find the optimal auction. Illustrates who gets the object for each pair of valuations graphically.

Problem 2.

This is a hard problem. Consider a setting with a seller and a buyer, whose valuations v_1 and v_2 are drawn independently from the uniform distribution on [0, 1]. Given the seller's valuation, the optimal mechanism to sell the object is to post the price of $p = (1+v_1)/2$ and to sell the object only if the buyer is willing to pay that price. However, what if the seller has an opportunity to design a mechanism before he learns his valuation? Prove that the mechanism above is still optimal.

Nash bargaining.

Problem 3.

Let c(S) be the Nash bargaining solution relative to the disagreement point (0, 0). Prove or find a counterexample to the following statement: if S is a subset of S′, then c(S′) is at least as good as c(S) to both players. If you find a counterexample, please illustrate it graphically. If you find a proof, please be concise.

Problem 4.

This problem is based on Nash (1953). Consider a bargaining situation is define by a convex set B of agreement possibilities, a set of threats for each player A_1 and A_2 and a mapping $u : A_1 \times A_2 \to B$. The bargaining outcome is determined by a game, in which the players choose threats (mixed actions) $t_1 \in \Delta(A_1)$ and $t_2 \in \Delta(A_2)$, and the outcome from B is determined by the Nash bargaining solution relative to the disagreement point

$u(t_1, t_2) = (u_1(t_1, t_2), u_2(t_1, t_2))$. The Nash bargaining solution $(v_1, v_2) \in B$ relative to this disagreement point maximizes $(v_1 - u_1(t_1, t_2))(v_2 - u_2(t_1, t_2))$. The *Nash bargaining solution with threats* is determined by a mixed strategy Nash equilibrium of this game.

Show that the Nash bargaining solution with threats satisfies the following properties.

(a) The solution (v_1, v_2) is Pareto efficient.

(b) For each bargaining game (B, A_1, A_2, u_1, u_2) the solution $(v_1, v_2) \in B$ is unique.

(c) Consider two bargaining games (B, A_1, A_2, u_1, u_2) and (B', A_1, A_2, u_1, u_2). If (v_1, v_2), the solution of the first game, is an element of B' and $B' \subset B$, then (v_1, v_2) is also chosen from B.

(d) A restriction of the set of strategies available to a player cannot increase the value to him of the game.

Corporate Finance.

Problem 5.

Consider the setting of Bolton and Scharfstein (1990), except with a modification that the entrepreneur has all the bargaining power and the investors act competitively. As before, in each of two periods, the firm needs F of outside funding to operate. If it operates, it gets cash flows π_1 or π_2 with probabilities θ and $1 - \theta$. Cash flows are iid and $\pi_1 < F < \pi_2$. The manager can conceal and divert cash flows, but the investors can force the manager to pay at least π_1. The residual $\pi_2 - \pi_1$ is the non-verifiable component, which the manager can give to the investors only if he has contractual incentives to do so. With competitive investors, the problem is to maximize the manager's expected payoff, subject to the constraint that the investors at least break even:

$$\max_{R_i, \beta_i, R^i} \theta[\pi_1 - R_1 + \beta_1(\bar{\pi} - R^1)] + (1-\theta)[\pi_2 - R_2 + \beta_2(\bar{\pi} - R^2)]$$

s.t.

$$-F + \theta[R_1 + \beta_1(R^1 - F)] + (1-\theta)[R_2 + \beta_2(R^2 - F)] \geq 0 \qquad \text{(investor breaks even)}$$

$$\pi_2 - R_2 + \beta_2(\bar{\pi} - R^2) \geq \pi_2 - R_1 + \beta_1(\bar{\pi} - R^1), \qquad \text{(truth-telling)}$$

$$\pi_i \geq R_i,\ \pi_1 \geq R^i.$$

(a) Prove that there exists an optimal contract with $R^1 = R^2 = \pi_1$?

(b) Show that any contract that gives positive profit to the investor can be modified to transfer the investor's profit to the manager.

(c) Is it true that $\beta_2 = 1$ in the optimal contract?

(d) Show that $R_1 = \pi_1$ in the optimal contract?

(e) Show that the truth-telling constraint is binding.

(f) What is the optimal contract?

(g) Under what conditions does a feasible contract exist? Compare this condition to one in the case when the investor is a monopolist. Interpret.

<u>Problem 6.</u>

Again, consider the setting of Bolton and Scharfstein (1990) with competitive investors, except assume this time that the agent also initially has cash $Y \in [0, 2(F - \pi_1)]$, which he can contribute to the project at time 0.

(a) Find R_1, R_2, R^1, R^2, β_1 and β_2 in the optimal contract. (Assume that the investors may be given the ability to disallow the agent to run the project even if he has enough cash to invest).

(b) Under what conditions does a feasible contract exist?

(c) Find the manager's expected payoff as a function of Y. Compare the derivative of his payoff with respect to Y with 1. Interpret.