Economics 209A
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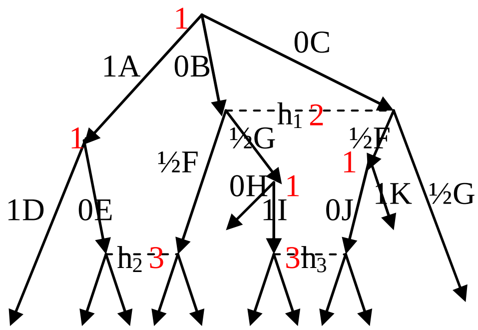
Solutions to Problem Set 2

Problem 1.

Consider the following figure, which illustrates partial information about the game (game tree, information sets, actions and strategies, but not players or payoffs).



(a) What is the minimal number of players in this game, for which all assumptions of Section 2 of Kreps and Wilson (1982) hold? Please justify your answer by showing how players can be assigned to nodes and information sets to ensure that all the assumptions hold, and explain why this cannot be done with fewer players.

Answer: 3 players (see figure above). At least 3 different players are required for the initial node and information sets h_1 and h_3.

(b) Please characterize the set of beliefs in information sets h_1 and h_3 which are consistent with the players' strategies. If one represents each belief with a number [0, 1], with 0 corresponding to a belief that assigns probability 1 to a left node, then the set of consistent beliefs on information sets h_1 and h_3 can be represented graphically by points inside a square. Please represent your answer graphically this way.

Denote by π the strategy profile in the figure. Consider a sequence of totally mixed strategies π_n that converge to π. Then the beliefs on h_1 are $\lim_{n\to\infty} \pi_n(C)/(\pi_n(B)+\pi_n(C))$ and the beliefs on h_3 are $\lim_{n\to\infty} \pi_n(J)\,\pi_n(C)/(½\,\pi_n(B)+\pi_n(J)\,\pi_n(C))$. If

$$\lim_{n\to\infty} \pi_n(C)/(\pi_n(B)+\pi_n(C))<1,$$

it follows that the beliefs on h_3 put probability 1 on the left node. If $\lim_{n\to\infty}$ $\pi_n(C)/(\pi_n(B)+\pi_n(C))=1$, then one can generate any beliefs on h_3 by letting

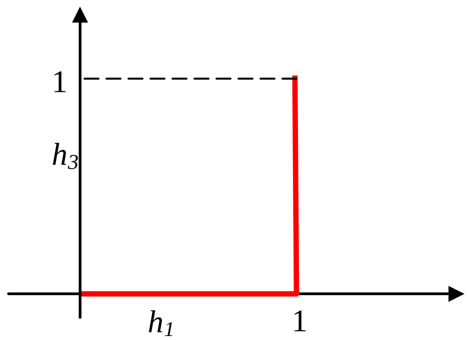
$$\pi_n(B) = p\varepsilon^2,\ \pi_n(C) = \varepsilon,\ \pi_n(J) = (1-p)\varepsilon$$

to put probability $p \in (0,1)$ on the left node in h_3,

$$\pi_n(B) = \varepsilon^3,\ \pi_n(C) = \varepsilon,\ \pi_n(J) = \varepsilon \qquad \text{to put probability 0 and}$$

$$\pi_n(B) = \varepsilon^2,\ \pi_n(C) = \varepsilon,\ \pi_n(J) = \varepsilon^2 \qquad \text{to put probability 1.}$$

We conclude that the set of consistent beliefs on information sets h_1 and h_3 looks as follows:



(c) For each combination of beliefs on sets h_1 and h_3, what is the set of beliefs on the information set h_2 that is consistent with strategies.

For each combination of beliefs on sets h_1 and h_3, any beliefs on h_2 are consistent with strategies. Indeed, one can let $\pi_n(E) = p\pi_n(B)/(2-2p)$ to generate probability $p \in (0,1)$ on the left node of h_2, let $\pi_n(E) = \pi_n(B)^2$ to generate probability 0 and $\pi_n(E) = \pi_n(B)^{1/2}$ to generate probability 1.

Remark: Part (a) was easy. Nobody got part (b) completely right. Please read the answer to part (b) especially carefully. Some people had problems with part (c) also.

<u>Problem 2.</u>

Consider the following game. There is a seller and many buyers. In each period, first the seller produces an item with quality either q = 0 or 1. It costs 0 to produce a low-quality item and 1 to produce a high quality item. A high-quality item has value v(1) = 3 to the buyers and a low-quality one has value v(0) = 1. After the item is produced, the seller holds a first-price auction, in which buyers simultaneously bid for the item. The item goes to the highest bidder, who pays his bid. Potential buyers do not know the true quality of the item when they submit bids, but learn about the true quality at the end of the period.

(a) If the game is repeated for 1 period, what happens in a SPE?

The seller produces a low-quality item and at least two buyers bid 1. The item is sold for the price of 1.

(b) If the game is repeated for N periods, what happens in a SPE?

By backward induction, the seller produces a low-quality item in each period (after all histories) at least two buyers bid 1, and the item is sold for the price of 1.

Now, suppose that the seller may be normal or behavioral. Normal seller can choose to produce an item of either quality, but the behavioral seller always produces an item of high quality. At the beginning the buyers hold a belief that the seller is behavioral with probability p_0, and the game is repeated for N periods.

(c) Characterize a sequential equilibrium of this game.

First, note that if the seller ever produces a low-quality item, then the buyers must believe that the seller is normal on the equilibrium path thereafter, and will offer a price of 1 in every period. Denote by p_n the probability that the buyers assign to the seller being behavioral at the beginning of period n if the seller has been producing high quality until then. Note that p_n must be a weakly increasing sequence. Denote by b_n the buyers bid in period n in case the seller has not revealed himself to be normal. In the following paragraphs, we analyze the game backwards after histories in which the seller has not revealed himself to be normal.

In the last period the seller will always produce a low-quality item and the buyers bid $b_N=1+2p_N$. In periods N-1 and N the seller earns a payoff of $b_{N-1} + 1$ if he produces an item of low quality, and a payoff of $b_{N-1} + 2p_N$ if he produces an item of the high quality. Therefore, if $p_{N-1} > ½$, the seller will produce high quality in period N-1 and if $p_{N-1} \le ½$, he will mix so that conditional upon producing high quality, the probability that he is normal rises to $p_N= ½$. If $p_{N-1} \le ½$, then the seller is normal and produces a high quality item with probability p_{N-1}. Therefore, the buyers bid $b_{N-1} = 1 + \min(2, 4p_{N-1})$ and the seller's payoff in the last two periods is $2 + 4p_{N-1}$ if $p_{N-1} \le ½$ and $3 + 2p_{N-1}$ if $p_{N-1} > ½$.

From this analysis, we form the following conjecture about the equilibrium behavior, which we can prove by backward induction. In period n, if $p_n \ge 2^{n-N}$ the seller of normal type produces an item of high quality, the buyers bid 3 and $p_{n+1} = p_n$. If $p_n < 2^{n-N}$, the seller (normal or behavioral) produces a high-quality item with probability $2^{n-N}p_n$ and the buyers bid $b_n + 1 + 2^{n-N+1}p_n$. If the seller happens to produce high quality, the buyers' belief in the next period becomes $p_{n+1} = 2^{n-N}$ and the buyers bid $b_{n+1}= 2$. It is easy to verify that if the seller is supposed to mix, he is indifferent between producing a low or a high quality item because $b_n +1+1\ldots = (b_n -1) + 2 + 1\ldots$ If the seller is supposed to produce a high-quality item in period n because $p_n \ge 2^{n-N}$, then he prefers to do so because $(3-1)+(b_{n+1}-1)+(b_{n+2}-1) \ldots + b_m + 1 \ldots \ge 3 + 1 + 1 + \ldots$ and $b_k \ge 2$ for all $k \ge n$.

Remark: Parts (a) and (b) are straightforward, but (c) proved to be hard. There was only one completely correct answer to part (c), although everybody had a sense of what goes on.

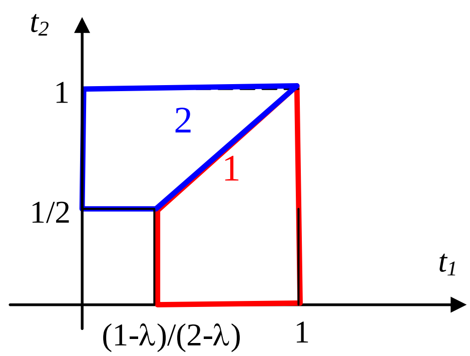
<u>Problem 3.</u>

(a) Find an optimal auction in the setting of Problem 2 on the previous problem set.

To derive an optimal auction, we rely on the proof of Lemma 3 from Myerson (1981), taking into account that the seller also gets ½ the utility of bidder 1. The seller's expected payoff is

$$\int_T \left(\left(t_1 - (1-\lambda)\frac{1-F_1(t_1)}{f_1(t_1)} \right) p_1(t) + \left(t_2 - \frac{1-F_2(t_2)}{f_2(t_2)} \right) p_2(t) \right) f(t)dt - (1-\lambda)U_1(p,x,0) - U_2(p,x,0)$$

To maximize his expected payoff, the seller must set $U_1(p,x,0) = U_2(p,x,0) = 0$. *Note that* $\frac{1-F_1(t_1)}{f_1(t_1)} = 1 - t_1$. *Therefore, to determine the winner in an optimal auction, the seller must compare* $(2-\lambda)t_1 - (1-\lambda)$, $2t_2 - 1$ *and* 0. *The following graph illustrates who gets the object in an optimal auction, as a function of valuations*



One way to define payments to make this into a truth-telling mechanism is to let x_1 *and* x_2 *be defined as in (4.8) of Myerson (1981):*

$$x_1 = \begin{cases} 0 & \text{if } t_1 < \max(\frac{1-\lambda}{2-\lambda}, \frac{2}{2-\lambda}t_2 - \frac{\lambda}{2-\lambda}) \\ \max(\frac{1-\lambda}{2-\lambda}, \frac{2}{2-\lambda}t_2 - \frac{\lambda}{2-\lambda}) & \text{otherwise} \end{cases}$$

and

$$x_2 = \begin{cases} 0 & \text{if } t_2 < \max(\frac{1}{2}, \frac{2-\lambda}{2}t_1 + \frac{\lambda}{2}) \\ \max(\frac{1}{2}, \frac{2-\lambda}{2}t_1 + \frac{\lambda}{2}) & \text{otherwise} \end{cases}$$

(b) Compute the seller's expected utility.

$$\int_T \Big(\big((2-\lambda)t_1 - (1-\lambda) \big) p_1(t) + \big(2t_2 - 1 \big) p_2(t) \Big) f(t) dt =$$

$$\int_{\frac{1-\lambda}{2-\lambda}}^{1} \int_{0}^{\frac{2-\lambda}{2}t_1+\frac{\lambda}{2}} \big((2-\lambda)t_1 - (1-\lambda) \big) dt_2 dt_1 + \int_{\frac{1}{2}}^{1} \int_{0}^{\frac{2}{2-\lambda}t_2-\frac{\lambda}{2-\lambda}} \big(2t_2 - 1 \big) dt_1 dt_2 =$$

$$\int_{\frac{1-\lambda}{2-\lambda}}^{1} \big((2-\lambda)t_1 + \lambda - 1 \big) \big(\tfrac{2-\lambda}{2} t_1 + \tfrac{\lambda}{2} \big) dt_1 + \int_{\frac{1}{2}}^{1} \big(\tfrac{2}{2-\lambda} t_2 - \tfrac{\lambda}{2-\lambda} \big) \big(2t_2 - 1 \big) dt_2 =$$

$$\int_{\frac{1-\lambda}{2-\lambda}}^{1} \Big(\tfrac{(2-\lambda)^2}{2} t_1{}^2 + \tfrac{(2\lambda-1)(2-\lambda)}{2} t_1 + \tfrac{\lambda(\lambda-1)}{2} \Big) dt_1 + \int_{\frac{1}{2}}^{1} \Big(\tfrac{4}{2-\lambda} t_2{}^2 - \tfrac{2(1+\lambda)}{2-\lambda} t_2 + \tfrac{\lambda}{2-\lambda} \Big) dt_2 =$$

$$\Big(\tfrac{(2-\lambda)^2}{6} (1 - \tfrac{(1-\lambda)^3}{(2-\lambda)^3}) + \tfrac{(2\lambda-1)(2-\lambda)}{4} (1 - \tfrac{(1-\lambda)^2}{(2-\lambda)^2}) + \tfrac{\lambda(\lambda-1)}{2} (1 - \tfrac{(1-\lambda)}{(2-\lambda)}) \Big) +$$

$$\Big(\tfrac{4}{2-\lambda} \tfrac{1}{3} \tfrac{7}{8} - \tfrac{(1+\lambda)}{2-\lambda} \tfrac{3}{4} + \tfrac{\lambda}{2-\lambda} \tfrac{1}{2} \Big) =$$

$$\Big(\tfrac{(2-\lambda)^2}{6} - \tfrac{1}{6} \tfrac{(1-\lambda)^3}{(2-\lambda)} + \tfrac{(2\lambda-1)(2-\lambda)}{4} - \tfrac{(2\lambda-1)}{4} \tfrac{(1-\lambda)^2}{(2-\lambda)} + \tfrac{\lambda(\lambda-1)}{2} + \tfrac{\lambda(\lambda-1)^2}{2} \tfrac{1}{(2-\lambda)} \Big) + \Big(\tfrac{-4-3\lambda}{2-\lambda} \tfrac{1}{12} \Big) =$$

$$\tfrac{1}{12} \tfrac{1-2\lambda+\lambda^2+2\lambda-4\lambda^2+2\lambda^3}{(2-\lambda)} + \tfrac{2+2\lambda^2+\lambda}{12} + \tfrac{5-3\lambda}{2-\lambda} \tfrac{1}{12} = \tfrac{1}{12} \tfrac{6-3\lambda-3\lambda^2+2\lambda^3}{2-\lambda} + \tfrac{2+2\lambda^2+\lambda}{12} = \tfrac{1}{12} \tfrac{\lambda^2}{2-\lambda} + \tfrac{5+\lambda}{12}$$

Remark. The most common incorrect approach was to try to prove that the auction suggested in problem set 1 is optimal. It is not. A couple of people had a correct approach and correctly found the expression for the seller's payoff. However, at that point nobody realized that the seller may keep the plant for some values of t_1 and t_2. I am not 100% sure about my algebra in part (b) because I could not check it against anybody.

Problem 4.

Suppose that there is a continuum of firms in the market with types t ∈ [c, d]. A firm of type t generates cash flows x in period 1 uniformly distributed on the interval [0, t]. In period 0 the manager of the firm, who knows the firm's type t but not the future cash flows, issues debt with face value F∈[0, ∞). The manager's compensation is given by

$$M = \gamma_0 V_0 - \gamma_1 \max(F - x, 0)$$

where V_0 is the market's perception in period 0 of the firm's expected cash flows, which is based on the publicly observable debt level F, max(F - x, 0) is the extent of bankruptcy in period 1 when cash flows x are realized, and γ_0 and γ_1 are positive constants. In period 0 the manager of each type chooses debt level F to maximize expected utility. Find the fully separating equilibrium. Specified what restrictions on parameters you imposed for this separating equilibrium.

Denote by T(F) the market's inference about the firm's type when the manager chooses debt level F. When F ∈ [0, t], the expected utility of the manager of type t is given by

$$\gamma_0/2\ T(F) - \gamma_1 \int_0^F (F - x) \tfrac{1}{t} dx = \gamma_0/2\ T(F) - \gamma_1\ F^2/(2t)$$

The first order condition implies $\gamma_0 T'(F) - \gamma_1 F/t = 0$. *Also, in a separating equilibrium we must have t = T(F), so we obtain an ODE for T:*

$$\gamma_0/2\ T'(F)\ T(F) = \gamma_1 F$$
$$2T'(F)\ T(F) = (T(F)^2)' = 4\gamma_1 F/\gamma_0$$
$$T(F)^2 = 2\gamma_1 F^2/\gamma_0 + \text{const}$$

The appropriate boundary condition for this equation is T(0) = c, so the separating equilibrium is defined by $T(F) = \sqrt{2\gamma_1 F^2 / \gamma_0 + c^2}$. *The manager of type t chooses debt level* $F(t) = \sqrt{\frac{\gamma_0}{2\gamma_1}(t^2 - c^2)}$. *We need to assume that* $\sqrt{\frac{\gamma_0}{2\gamma_1}(d^2 - c^2)} \le d$, *so F(t)*$\in$*[0,t] for each type t.*

Remark: About half the people got this problem right.