Economics 209A
Professor Yuliy Sannikov

Solutions to Problem Set 1.

Directions: Make every attempt to do each problem on your own. If you need help, please feel free to consult papers or discuss the problems with your classmates. On your solutions, please acknowledge any references you used or help you got. For example, "I discussed this problem with so and so."

In your formal proofs, please be concise.

Problem 1.

Consider the following game related to the war of attrition discussed in class. Two players initially make demands (2,1) and (1,2) about how to split $3, and play the war of attrition. Both players discount future payoffs at the common rate r. However, unlike in the standard war of attrition, player 2 may be a behavioral type with probability $p \in (0,1)$ or a normal type with probability 1-p. The behavioral type is not able to concede. The structure of the game is common knowledge, but only player 2 knows whether he is really normal or behavioral. The purpose of this problem is to characterize all mixed strategy Bayesian Nash equilibria. Denote by F_1 the CDF of concession times of player 1, and by F_2 the CDF of concession times of the *normal* type of player 2. Denote by T the time when the play ends for sure, i.e. $T = \min_t \{F_1(t) = 1\}$.

(a) Prove that $\min_t \{F_2(t) = 1\} \geq T$

Suppose on the contrary that $F_2(t) = 1$ and $T > t$.. Then after time t player 1 knows that player 2 will never concede because he is behavioral for sure. Therefore, it is optimal for player 1 to concede immediately, which contradicts $T > t$.

(b) Prove that F_1 and F_2 may not have atoms on (0, T].

Suppose that F_i has an atom at time $t \in (0, T]$ with weight p. Then it is strictly better for player j to concede immediately after time t than at time t-ε if

$$1 < 2pe^{-r\varepsilon} + (1-p)e^{-r\varepsilon} \Leftrightarrow \varepsilon < \log(1 + p)/r.$$

Knowing that player j does not concede on the interval $(t - \log(1 + p)/r, t]$, it is strictly better for player i to concede at time $t - \log(1 + p)/(2r)$ than at time t, which is a contradiction.

(c) Prove that there is no subinterval in (0, T) where both F_1 and F_2 are flat.

Suppose that F_1 and F_2 are flat on the interval $[t', t'')$ and player i starts conceding again after time t'', in the sense that $F_i(t''+\varepsilon) > F_i(t')$ for all $\varepsilon > 0$. Then from the continuity of F_j (by (b) F_j has no atoms) it follows that for any $\delta>0$ there exists $\varepsilon(\delta)>0$ such that $F_j(t''+\varepsilon) < F_i(t')+\delta$. Let us show that for sufficiently small δ, player i's payoff from conceding at time $(t'+t'')/2$ is strictly better than his payoff from conceding at time $t''+\varepsilon$ for all $\varepsilon \in [0, \varepsilon(\delta))$, which leads to a contradiction. We need to compare

$$e^{-r(t''+t')/2} \qquad \text{and} \qquad e^{-rt''}(1+\delta)$$

where the second expression is an upper bound on player i's payoff if he concedes at a time $t\in[t'', t''+\varepsilon(\delta))$. Clearly, δ can be chosen sufficiently small so $e^{-rt''}(1+\delta)< e^{-r(t''+t')/2}$.

(d) Prove that there is no subinterval in (0, T) where F_1 or F_2 alone is flat.

Suppose that F_i alone is flat on the interval (t', t''). Then part (c) implies that F_j must be strictly increasing. However, because player i does not concede within the interval (t', t''), it is strictly better for player j to concede earlier in the interval than later. Therefore, F_j cannot be strictly increasing on (t', t''), a contradiction.

(e) Find all Bayesian Nash equilibria, and justify your logic.

The expected payoff of the normal type of player 2 must be constant for all concession times $t \in (0, T]$:

$$\int_0^t 2e^{-rs}dF_1(s) + e^{-rt}(1-F_1(t)) = const \qquad \Rightarrow \qquad f_1(t) = r(1-F_1(t)).$$

From this we conclude that if $F_1(0) < 1$, then F_1 never reaches 1 in finite time and $T = \infty$. In case $T = \infty$, the expected payoff of player 1 must be constant for all concession times $t \in (0, \infty)$:

$$(1-p)\int_0^t 2e^{-rs}dF_2(s) + e^{-rt}(1-(1-p)F_2(t)) = const \qquad \Rightarrow \qquad (1-p)f_2(t) = r(1-(1-p)F_2(t))$$

This ODE has solutions $F_2(t) = \dfrac{1}{1-p} - ke^{-rt}$. *Any such function F_2 becomes greater than 1 for large t, which leads to a contradiction. We conclude that $F_1(0) = 1$ is the only possibility, so that in any Bayesian Nash equilibrium player 1 concedes for sure at time 0. Player 2 can have any density of concession times that satisfies*

$$(1-p)\int_0^t 2e^{-rs}dF_2(s) + e^{-rt}(1-(1-p)F_2(t)) \le 1.$$

Note: For this problem, Abreu and Gul (2000) is helpful.

Problem 2.

A seller is auctioning a small plant, and two companies are participating in an auction. The valuations v_1 and v_2 of the two companies are drawn independently from the uniform distribution on [0, 1]. The seller owns a fraction λ of the first company, and therefore gets λ of its profit. If company 1 obtains the plant and pays b_1, then the seller's payoff is $b_1 + \lambda(v_1 - b_1)$. If company 2 obtains the plant and pays b_2, then the seller's payoff is b_2. Because of this asymmetry, the seller would like to design an auction that favors company 1. Therefore, instead of comparing the bids to determine the winner, the seller chooses to compare ab_1 and b_2, where $a > 1$ is announced at the beginning of the auction. The winning company pays its bid.

(a) Find an equilibrium, in which the bidding functions are of the form

$$\sigma_1(v_1) = \begin{cases} kv_1 & v_1 \le v^* \\ kv^* & v_1 \ge v^* \end{cases} \quad \text{for company 1 and}$$

$$\sigma_2(v_2) = lv_2 \quad \text{for company 2,}$$

with $akv^* = l$.

Solution. *If company 1 bids b_1, it wins the item when $ab_1 \ge lv_2$, which happens with probability* $\min(ab_1/l, 1)$. *Because its payoff is decreasing in b_1, the equilibrium bid b_1 is always within the range $[0, l/a]$. The company's expected payoff $(v_1-b_1)ab_1/l$ is maximized when $b_1 = \min(v_1/2, l/a)$. We conclude that $k = ½$ and $v^*=2l/a$. For player 2, the expected payoff is $2(v_2-b_2)b_2/a$, which is maximized when $b_2 = v_2/2$. We conclude that $l = ½$ and $v^*=1/a$.*

(b) Compute the seller's expected payoff from this equilibrium.

Solution. *The seller's expected payoff can be computed as follows:*

$$\int_0^1 \int_0^{v_2/a} \frac{v_2}{2} dv_1 dv_2 + \int_0^1 \int_{v_2/a}^{1/a} \left(\frac{v_1}{2} + \lambda\frac{v_1}{2}\right) dv_1 dv_2 + \int_{1/a}^{1} \left(\frac{1}{2a} + \lambda\left(v_1 - \frac{1}{2a}\right)\right) dv_1 =$$

$$\frac{1}{6a} + (1+\lambda)\frac{1}{2a}\frac{1}{3a} + (1-1/a)\left((1-\lambda)\frac{1}{2a} + \lambda(1+1/a)/2\right) = -\frac{1}{3a^2} + \frac{2}{3a} - \frac{\lambda}{2a} + \frac{\lambda}{6a^2} + \frac{\lambda}{2}$$

(c) For what value of a is the seller's expected payoff maximized in an auction with this structure?

Solution. *Differentiating with respect to 1/a we obtain* $a = \dfrac{2-\lambda}{2-3/2\lambda}$.