

# Toward a Theory of Discounted Repeated Games with Imperfect Monitoring

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# Summary

- Studies repeated games with imperfect monitoring.
- Set  $V$  of pure strategy sequential equilibrium (S.E) payoffs found by solving a collection of static problems: how to generate a given payoff vector with a current action profile and supporting vectors of future payoffs.
- Computational procedure to find  $V$ .
- Other results:
  - *Sufficient* conditions to look only to extreme continuation payoffs
  - *Necessary* conditions
- Analyzes ***asymmetric*** equilibria, and complements APS (1986), which focuses on ***symmetric*** equilibria.

# The Stage Game G

- $N$  players
- Each player  $i \in N$  chooses a *private* (unobservable) action  $q_i \in S_i$  (finite set)
- Publicly observable signal  $p$  with density  $g(\cdot, q)$ , where  $q \in S = S_1 \times \dots \times S_N$
- Player  $i$  gets payoff  $\pi_i(p, q_i)$ .

- Define

$$\Pi_i(q) = \int_{\Omega} \pi_i(p, q_i) g(p; q) dp$$

# The Repeated Game $G^\infty(\delta)$

- Game  $G$  repeated with discount factor  $\delta$
- At time  $t$  player  $i$  knows
  - Public history  $p_1 p_2 \dots p_{t-1}$
  - Her past actions  $q_1 q_2 \dots q_{t-1}$
- A strategy of player  $i$  is a sequence  $\{\sigma_i(t)\}_{t=1,2,\dots}$

$$\sigma_i(1) \in S_i, \quad \sigma_i(t) : \Omega^{t-1} \times S_i^{t-1} \rightarrow S_i \quad \text{for } t > 1$$

- $v(\sigma) = (v_1(\sigma), \dots, v_N(\sigma))$ : present discounted value of stochastic payoff stream under strategy profile  $\sigma$
- Assume:  $g(p; q) > 0 \quad \forall p, q$ ;  $G$  has a pure Nash eq.
- Wish to characterize  $V = \{v(\sigma) : \sigma \text{ is a S.E.}\}$

# Notation Drill

- **Notation:**  $q, \sigma, \pi, v, g, \rho$
- **What it means:** signal, payoff function, probability density of signal, value of a strategy profile, vector of actions, strategy profile

# Begin Analysis

- Wish to characterize  $V = \{v(\sigma) : \sigma \text{ is a S.E.}\}$
- Can restrict attention to *public* strategies:

$$\sigma_i(1) \in S_i, \quad \sigma_i(t) : \Omega^{t-1} \rightarrow S_i \quad \text{for } t > 1$$

- Why?
- Any S.E.  $\sigma$  induces a continuation profile  $\sigma|_p$  for all  $p \in \Omega$
- $\sigma|_p$  is common knowledge and is itself a S.E.
- Therefore, for all signals  $p$ ,  $v(\sigma|_p) \in V$

# Pairs (q,u) admissible w.r.t. V

- For equilibrium  $\sigma$  let  $q$  be a first-period action profile and  $u(p) = v(\sigma|_p)$  for  $p \in \Omega$ . Then

$$v(\sigma) = E(q; u) = (E_1(q; u), \dots, E_N(q; u))$$

$$E_i(q; u) = \Pi_i(q) + \delta \int_{\Omega} u_i(p) g(p; q) dp$$

- **Definition.** A pair  $(q, u)$  is *admissible with respect to V* if for all  $p \in \Omega$ ,  $u(p) \in V$  and for all  $i \in N$ ,

$$q_i \in \operatorname{argmax} E(q_i, q_{-i}; u).$$

- For any S.E.  $\sigma$ , the associated pair  $(q, u)$  is admissible with respect to V.
- We can say when a pair  $(q, u)$  is admissible w.r.t. V independently of whether it is a part of S.E. or not

# Factorization

- Denote by  $B(V) = \{E(q,u) \mid (q,u) \text{ admissible w.r.t. } V\}$
- Recall: 
$$E(q;u) = \Pi(q) + \delta \int_{\Omega} u(p)g(p;q)dp$$
- Because any  $v \in V$  has an associated S.E. and an admissible pair  $(q, u)$ ,  $V \subseteq B(V)$
- **Conversely:**  $\forall v \in B(V)$  take adm. pair  $(q,u)$  with value  $v$ , choose a S.E.  $\sigma(p)$  with value  $u(p)$  for all  $p \in \Omega$ .
- Construct S.E.  $\sigma$  with value  $v$  as follows:  
q (current actions)  $\rightarrow$  p (signal)  $\rightarrow$   $\sigma(p)$  S.E. of “subtree”
- Thus,  $v \in B(V)$  and so  $B(V) \subseteq V$

**Theorem 2: (Factorization)  $V = B(V)$**

# Self-Generation

- **Definition.** A pair  $(q, u)$  is *admissible with respect to*  $W$  if  $\forall p \in \Omega, u(p) \in W$  and  $i \in N$ ,  
$$q_i \in \operatorname{argmax} E(q_i, q_{-i}; u).$$
- Denote  $B(W) = \{E(q, u) \mid (q, u) \text{ admissible w.r.t. } W\}$

**Theorem 1.** (Self-Generation) For any bounded Borel set  $W \subset \mathbb{R}^N$ , if  $W$  is self-generating then  $B(W) \subseteq V$ .

# Self-Generation

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“Proof.” Choose  $v \in W$ . Let us construct a S.E. w/ value  $v$ .

- $\forall w \in W$  select a pair  $(q_w, u_w)$  with value  $w$ .
- Draw the tree of public histories.
- Recursively, write a value and action profile in each node:
  - Start from  $v$  in the initial node.
  - If  $w$  is written in a given node, write action  $q_w$  in that node and values  $u_w(p)$  in successive nodes.
- These action profiles generate a S.E. over the entire tree.
- Therefore,  $v \in V$ . QED

# Extreme Reward Functions.

- What are extreme points of a set?
- **Theorem 3.** Let  $W \subseteq \mathbb{R}^N$  be compact. Suppose  $(q, \hat{u})$  is admissible with respect to  $\text{co } W$ . Then there is a function  $\bar{u} : \Omega \rightarrow \text{ext } W$  such that  $(q, \bar{u})$  is admissible with respect to  $W$  and

$$E(q; \bar{u}) = E(q; \hat{u})$$

- Critical assumption: distribution of signals is nonatomic.
- Exercise:  $\Omega = [0, 2]$ ;  $g(\cdot, q')$  constant on  $[0, 1]$  and  $(1, 2]$  for all  $q'$ . Take a reward function  $\hat{u} : \Omega \rightarrow \text{co } W$ . Construct an equivalent reward function  $\bar{u} : \Omega \rightarrow \text{ext } W$ .
- **Corollary.** If  $W$  is compact, then  $B(W) = B(\text{co } W)$ .

# Compactness, Monotonicity

- **Lemma 1.** The operator  $B$  is monotone and preserves compactness:
  - (i) if  $W \subseteq W'$  then  $B(W) \subseteq B(W')$
  - (ii) if  $W$  is compact then  $B(W)$  is compact
- **Theorem 4.**  $V$  is compact.
- *Proof.*  $V = B(V) \subseteq B(\text{cl}(V))$ .  $B(\text{cl}(V))$  is compact and contains  $V$ , so it must contain  $\text{cl}(V)$ . This means  $\text{cl}(V)$  is self-generating, so  $\text{cl}(V) \subseteq V$ . Thus  $V = \text{cl}(V)$ . QED

# Computation

- **Algorithm.** Start with a compact set  $W \subset \mathbb{R}^N$  s. t.  $V \subseteq B(W_0) \subseteq W_0$ . Compute  $W_n = B(W_{n-1})$ ,  $n = 1, 2, \dots$  (monotonically decreasing). The limit of this process is  $V$ , intersection of all  $W_n$ .
- *Sketch of proof.*  $\{W_n\}$  is decreasing and  $W_n$  contains  $V$  for all  $n$ . How do we know it converges to  $V$ , not a larger set  $W_\infty$ ? We know because  $W_\infty$  is self-generating.
- Any  $w$  in  $W_\infty$  is also in all  $W_n$ .  $\forall n$  there is a pair  $(q_n, u_n)$  with value  $w$ . Sequence  $\{(q_n, u_n)\}$  must have a convergent subsequence. The limit pair  $(q, u)$  will also have value  $w$ , and will be admissible w.r.t. all  $W_n$ , and  $W_\infty$  also. Therefore,  $w$  is generated by  $W_\infty$  so  $W_\infty \subseteq B(W_\infty)$ . QED