Repeated Games.

Two-player game with simultaneous moves:

$$G \;=\; \{N = \{1,2\},\;\; (\mathcal{A}^i)_{i=1,2},\;\; (g_i)_{i=1,2}\}.$$

$N =$ $\leftarrow$ set of players

$\mathcal{A}^i$ $\leftarrow$ set of actions of player i

$g_i : \mathcal{A}^1 \times \mathcal{A}^2 \to \Re$ $\leftarrow$ payoff function of player i

Examples: Prisoners' Dilemma

	C	D
C	1, 1	-1, 2
D	2, -1	0, 0

Duopoly

	L	M	H
L	10,10	3,15	0,7
M	15,3	7,7	-4,5
H	7,0	5,-4	-15,-15

Nash equilibrium: pair of actions $a_1 \in \mathcal{A}^1$ and $a_2 \in \mathcal{A}^2$ such that a_1 is a best response to a_2 and vice versa

Repeated Prisoners' Dilemma.

Repeated stage game

	C	D
C	1, 1	-1, 2
D	2, -1	0, 0

Payoff of player i

$$\sum_{t=1}^{\infty} \delta^{t-1} g_i(A_t)$$

Example: $\underbrace{(C,C),(C,C),(D,C),}_{\text{history}}(C,D),(D,C),(D,C),(D,C)\ldots$

Player 1 gets: $1+\delta+2\delta^2-\delta^3+2\delta^4+0\delta^4+0\delta^5\ldots$

Player 2 gets: $1+\delta-\delta^2+2\delta^3-\delta^4+0\delta^4+0\delta^5\ldots$

Strategy: specifies an action after all histories

Examples:

1. always play D
2. play what the opponent played in previous period
3. play C until the opponent plays D, then play D

Equilibrium in a Repeated Game.

pair of strategies, such that each player acts optimally in response to her opponent after every history

Examples:

(1) Both players always play D

- repetition of the Nash equilibrium of the stage game

(2) Both players follow this strategy:

play C until the opponent plays D, then play D

How to check subgame perfection:

One-Stage Deviation Principle: In a repeated game, a combination of players' strategies s is a SPE if and only if no player can gain by deviating in a single stage, and confirming to s thereafter.

Proposition. If $\delta \geq 1/2$, the repeated prisoners' dilemma has a SPE in which (C,C) is played in every period.

Proof. We show that the following two strategies generate a SPE when $\delta \geq 1/2$: play C until the opponent plays D, then play D.

On eq. path players cooperate forever, but if anybody defects (off-equilibrium action) then they switch to defection forever.

Need to check: deviation after any history not profitable

Off the equilibrium path: both players defect, neither player wants to deviate.

On the equilibrium path:

payoff if deviate: $2 + 0\delta + 0\delta^2 + \ldots = 2$

Payoff if cooperate: $1 + \delta + \delta^2 \ldots = \frac{1}{1-\delta}$

Deviation not profitable if $\delta \geq 1/2$. QED

Exercise: For what values of δ is cooperation possible in the following game:

	C	D
C	π, π	$-b, \pi + g$
D	$\pi + g, -b$	0,0

$\delta \geq \ldots$

Equilibria with Payoffs Worse than Nash.

	D	E
D	0, 0	-1, -1
E	-1, -1	-2, -2

Claim. When $\delta \geq 1/2$ there is a SPE that achieves payoff (-2, -2).

Proof. Let us construct a SPE with two regimes.

Regime 1: play (E, E)

Regime 2: play (D, D)

Start in regime 1.

Transitions: if both players choose (E, E) in regime 1, go to regime 2 in the next period. Otherwise stay in regime 1. Regime 2 is absorbing.

Check: neither player would deviate in regimes 1 or 2.

Regime 2: repetition of Nash

Regime 1: payoff if not deviate -2

if deviate once $-1 + -2\delta + 0\delta^2 + \dots$

This gives us a SPE if $\delta \geq 1/2$.

Why Payoffs Worse than Nash?

	C	D	E
C	1,1	-1, 2	-2, 1
D	2, -1	0, 0	-1, -1
E	1, -2	-1, -1	-2, -2

Exercise For what values of δ is it possible to have cooperation?

Why Payoffs Worse than Nash?

	C	D	E
C	1,1	-1, 2	-2, 1
D	2, -1	0, 0	-1, -1
E	1, -2	-1, -1	-2, -2

Claim. When $\delta \geq 1/3$ there is a SPE that achieves cooperation forever.

Proof. Let us construct a SPE with two regimes.

Regime 1: play (C, C)

Regime 2: play (E, E)

Start in regime 1.

Transitions: if both players choose (C, C) in regime 1, stay in regime 1 in the next period. Otherwise go to regime 2. In regime 2, if both players chose (E, E) then go to regime 1. Otherwise stay.

Payoff in regime 1: $1 + \delta + \delta^2 \ldots$

regime 2: $-2 + \delta + \delta^2 + \ldots$

Check: neither player would deviate in regimes 1 or 2.

Regime 2: $-2+\delta+\delta^2 \geq -1-2\delta+\delta^2+\delta^3 \quad \Leftrightarrow \quad \delta \geq 1/3$

Regime 1: $1+\delta+\delta^2 \ldots \geq 2-2\delta+\delta^2+\delta^3 \ldots \quad \Leftrightarrow \quad \delta \geq 1/3$