

Simultaneous-Move Games

Optional Reading: MWG 8: A, B, D

Definition: A simultaneous-move game is defined by:

- a set of players, $N = \{1, 2, \dots, n\}$
- a set of actions of each player A_i , $i \in N$
- payoff functions $g_i : A_1 \times A_2 \times \dots \times A_n \rightarrow R$

A *strategy* is a complete plan of actions of a player. In a simultaneous-move game, a strategy is just an action.

Prisoners' Dilemma, Strict Dominance

		Player 2	
		C	D
Player 1	C	1, 1	-1, 2
	D	2, -1	0, 0

$$A_1 = A_2 = \{C, D\}$$

$$g_1(C, C) = g_2(C, C) = 1$$

$$g_1(D, C) = 2, \quad g_2(D, C) = -1$$

$$g_1(C, D) = -1, \quad g_2(C, D) = 2$$

$$g_1(D, D) = g_2(D, D) = 0$$

Notation:

- $A = A_1 \times A_2 \dots A_n$ set of all combinations of *actions*, typical element $a = (a_1, \dots, a_n)$.
- $A_{-i} = A_1 \times \dots \times A_{i-1} \times A_{i+1} \dots A_n$ set of all combinations of the actions of player i 's opponents, typical element a_{-i}

Definitions:

- Strategy $a_i \in A_i$ of player i *strictly dominates* strategy $a_i' \in A_i$ if for all $c \in A_{-i}$, $g_i(a_i, c) > g_i(a_i', c)$.
- Strategy a_i is *strictly dominant* if it strictly dominates all other strategies.

Iterative Elimination of Strictly Dominated Strategies

	A	B
A	0, -2	-5, -1
B	-1, -5	-4, -4

Definitions:

- $a_i \in A_i$ weakly dominates $a_i' \in A_i$ if $\forall c \in A_{-i}$
 $g_i(a_i, c) \geq g_i(a_i', c)$ w/ strict inequality for at least one c
- a_i is weakly dominant if it weakly dominates all other strategies

Bertrand competition

- 2 firms.
- actions are prices $A_1 = A_2 = \{1, 2, 3\}$
- zero marginal costs, no fixed costs
- demand is $q = 12 - 2p$
- all q customers go to the firm that sets a lower price
(or split between firms if prices are equal)

Exercise: draw payoff matrix

Price	1	2	3
Quantity	10	8	6
Profit	10	16	18

Payoff matrix

		Firm 2		
		1	2	3
Firm 1	1	5, 5	10, 0	10, 0
	2	0, 10	8, 8	16, 0
	3	0, 10	0, 16	9, 9

Exercise: do iterative elimination of strictly dominated strategies.

Nash Equilibrium in Pure Strategies

Iterative elimination of dominated strategies does not always give a solution.

		Player 2	
		A	B
Player 1	A	1, -2	-6, -1
	B	-2, -5	-4, -4
	C	0, 0	-5, -1

Any strictly dominated strategies?

Definition:

- Strategy a_i is a *best response* to a_{-i} if a_i maximizes $g_i(\cdot, a_{-i})$.
- A strategy profile (a_1, \dots, a_n) such that a_i is a best response to a_{-i} for all $i \in N$ is a *Nash equilibrium*

		Player 2	
		A	B
Player 1	A	1, -2	-6, -1
	B	-2, -5	-4, -4
	C	0, 0	-5, -1

Coordination:

		Player 2	
		A	B
Player 1	A	1, 1	0, 0
	B	0, 0	1, 1

Risky Coordination:

		A	B
		1, 1	0, -1000
A			
B		-1000, 0	10, 10

Coordination 2:

		Player 2	
		A	B
Player 1	A	1, 1	0, 0
	B	0, 0	2, 2

Battle of the Sexes:

		Ballet	Football
		2, 1	0, 0
Ballet			
Football		0, 0	1, 2

Cournot Competition.

- Two firms set quantities q_1 and $q_2 \in [0, \infty)$.
- Market demand $P(q_1+q_2) = 12 - q_1+q_2$.
- Payoffs are $g_i(q_i, q_{-i}) = q_i P(q_1+q_2)$.

Exercise: Find Nash equilibria.

Matching Pennies:

		Player 2	
		Heads	Tails
Player 1	Heads	1, -1	-1, 1
	Tails	-1, 1	1, -1

- zero-sum game
- player 1 wants to match
- player 2 does not want to match

No Nash equilibria in pure strategies

Mixed Strategies, e.g. $\frac{1}{4}$ Heads + $\frac{3}{4}$ Tails

Definition: A *mixed strategy* of a player i is a probability distribution over A_i . We will denote a mixed strategy by σ_i , so that $\sigma_i(a_i)$ is the probability of a_i .

Payoffs can be defined over mixed strategies like this:

$$g_i(a_i, \sigma_{-i}) = \sum_{a_{-i} \in A_{-i}} \sigma_{-i}(a_{-i}) g(a_i, a_{-i})$$

$$g_i(\sigma_i, \sigma_{-i}) = \sum_{a_i \in A_i} \sigma_i(a_i) g(a_i, \sigma_{-i})$$

Concepts of strict dominance, iterative elimination of dom. strategies and Nash eq. extend to mixed strategies

		Player 2	
		A	B
Player 1	A	3, 3	3, 2
	B	0, 4	6, 1
	C	1, 5	4, 6

Matching Pennies

	Heads	Tails
Heads	1, -1	-1, 1
Tails	-1, 1	1, -1

In general, this 2x2 game has no pure strategy Nash eq.

	C	D
A	$a+b, x$	$c, x+y$
B	$a, z+w$	$c+d, z$

Exercise: find mixture between C & D that makes player 1 indifferent

Mixed Strategy Equilibria: Larger Games

Proposition: (σ_1, σ_2) is a mixed strategy Nash equilibrium if and only if for all $i \in N$

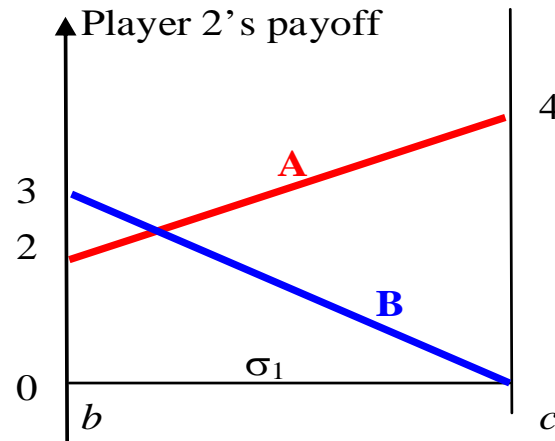
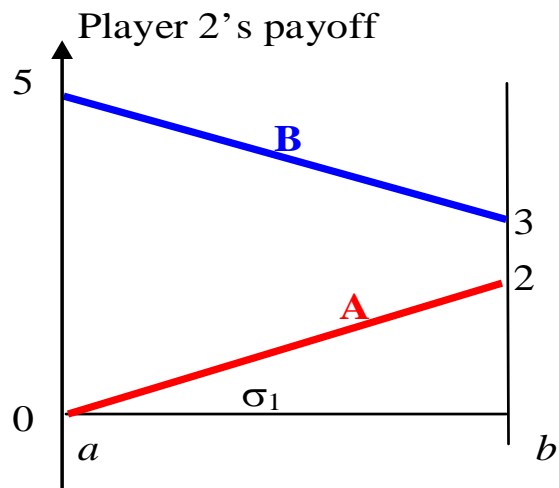
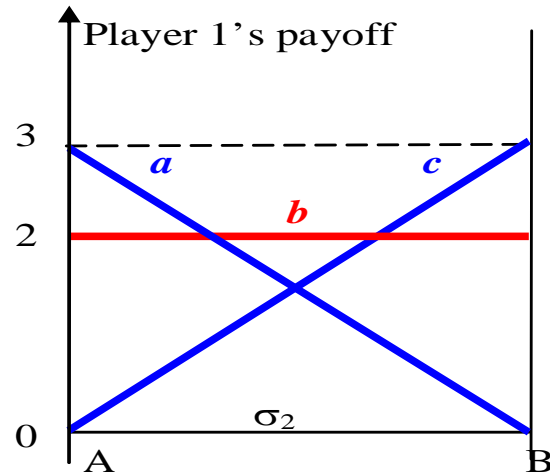
- $g_i(a_i, \sigma_{-i}) = g_i(\sigma_i, \sigma_{-i})$ for all a_i to which σ_i assigns positive probability
- $g_i(a_i, \sigma_{-i}) \leq g_i(\sigma_i, \sigma_{-i})$ for all a_i to which σ_i assigns probability zero

Intuition: a player must be indifferent between all actions that he uses in mixing, and must weakly prefer them to all other available actions

Exercise: Find Mixed Strategy Nash Equilibria

	A	B	C
<i>A</i>	3, 0	0, 5	0, 2
<i>B</i>	2, 2	2, 3	0, 0
<i>C</i>	0, 4	3, 0	0, 1

	A	B
A	3, 0	0, 5
B	2, 2	2, 3
C	0, 4	3, 0



Exercise: compute the probabilities explicitly

Exercise. A two-player game with finitely many actions is played repeatedly. Imagine the following kind of dynamics:

- In period 1, players choose a given pair of actions (a_1, a_2)
- In each *even* period thereafter, player 1 chooses the same action as in the previous period, and player 2 chooses a best response to that action (e.g. in period 2 player 2 chooses a best response to a_1).
- In each *odd* period, player 2 keeps the action from the previous period, while player 1 chooses a best response to that action

- (a) **in class:** If the game has a unique pure strategy Nash equilibrium, will the players always converge to it? Prove, or give a counterexample.
- (b) **homework:** If the game has a unique pair of strategies that survives iterative elimination of strictly dominated strategies, will the players converge to it? Prove, or give a counterexample.
- (c) **homework:** Suppose that players never converge. Is there always a mixed strategy Nash equilibrium that involves *only* the actions used along the path of play?