Autocorrelation Robust Tests with Good Size and Power

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Abstract. A new class of autocorrelation robust test statistics is introduced. The class of tests generalizes the Kiefer, Vogelsang, and Bunzel (2000) test in a manner analogous to Anderson and Darling's (1952) generalization of the Cramér-von Mises goodness of fit test. In a Gaussian location model, the error in rejection probability of the new tests is found to be $O\left(T^{-1}\right)$, where T denotes the sample size. Appropriately selected tests dominate the Kiefer, Vogelsang, and Bunzel (2000) test in terms of local asymptotic power.

1. Introduction

In many applications in time series econometrics, estimators that enjoy optimality properties in cross-sectional environments remain asymptotically normally distributed, albeit with a covariance matrix that depends on the autocovariance function of the data. A leading example is the OLS estimator in a linear regression model with exogenous regressors and an autocorrelated error term. The conventional approach in the literature is to base autocorrelation robust inference on Wald-type test statistics constructed by employing a standardization involving a consistent estimator of the asymptotic covariance matrix of an estimator of the parameter of interest (e.g. Robinson and Velasco (1997) and Wooldridge (1994)). While this approach often delivers inference procedures with certain asymptotic optimality properties, the finite sample size properties of these procedures has been found to be somewhat less than satisfactory in many cases (e.g. den Haan and Levin (1997)).

Kiefer, Vogelsang, and Bunzel (2000, hereafter denoted KVB) demonstrate that the size properties of Wald-type test statistics can be ameliorated if an inconsistent covariance matrix "estimator" is used and the critical values are adjusted to accommodate the randomness of the matrix employed in the standardization. Any size improvements (relative to conventional test statistics) achieved by employing inconsistent variance estimators necessarily come at the expense of local asymptotic power and there is a noticeable difference between the local asymptotic power properties of

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the KVB testing procedure and those of conventional procedures. As a consequence, it seems natural to ask whether it is possible to construct an autocorrelation robust inference procedure that matches the KVB procedure in terms of size properties and dominates it in terms of local asymptotic power.

The present paper makes three contributions. First, a new class of autocorrelation robust inference procedures is introduced. In a manner analogous to Anderson and Darling's (1952) generalization of the Cramér-von Mises goodness of fit test, the class of autocorrelation robust tests introduced here generalizes the KVB test by accommodating a weight function in the definition of the covariance matrix "estimator" used in the construction of the test statistic.

Second, the paper sheds new light on the size properties of the KVB test and its generalizations introduced herein. In a Gaussian location model, the error in rejection probability (ERP) of the new tests is found to be $O\left(T^{-1}\right)$ (where T denotes the sample size) when the critical value suggested by the first-order asymptotic distribution of the test statistic is used. This result is very encouraging, as the ERP of conventional procedures is no better than $O\left(T^{-1/2}\right)$ under similar circumstances (Velasco and Robinson (2001)). Indeed, in spite of the restrictive nature of the assumptions under which this higher-order asymptotic result is obtained, the rate $O\left(T^{-1}\right)$ is remarkable in view of the fact that existing results on the performance of bootstrap-based autocorrelation robust inference procedures indicate that even these procedures fail to achieve the same rate of convergence in the presence of nonparametric autocorrelation (Härdle, Horowitz, and Kreiss (2001)).

Finally, the paper characterizes the local asymptotic power properties of the new tests in a linear regression model under fairly general assumptions. It is shown that while all members of the new class of tests are inferior to conventional tests in terms of local asymptotic power, the shortcoming of the new tests can be made arbitrarily small by employing appropriately chosen weight functions. Although the finite-sample implications of this near-optimality result are unclear, the insights provided by the derivation of the result suggest how nontrivial power improvements over the KVB test can be achieved in sample sizes of empirical relevance. In other words, the constructive nature of the proof of the near-optimality result makes it possible to provide an affirmative answer to the power-related question posed at the end of the second paragraph.

In sum, this paper constructs an autocorrelation robust inference procedure that matches the KVB procedure in terms of higher-order size properties and dominates it in terms of local asymptotic power properties. The finite sample relevance of these asymptotic results is assessed by means of a small Monte Carlo experiment in which the test proposed in this paper is found to perform very well.

Section 2 introduces the basic model, states the assumptions under which formal results will be developed and introduces the new class of inference procedures. Section 3 studies higher-order size properties in a Gaussian location model, while local asymptotic power results are presented in Section 4. Section 5 investigates finite sample properties by means of a Monte Carlo experiment. Finally, Appendix A is devoted to the development of the asymptotic properties of the "estimators" employed in the testing procedure and proofs are collected in Appendix B.

2. PRELIMINARIES

The basic framework is that of KVB. Consider the linear regression model

$$y_t = x_t'\beta + u_t, \qquad t = 1, \ldots, T, \tag{1}$$

where x_t is a k-vector of regressors, β is a k-vector of parameters and u_t is an unobserved error term with $E(u_t|x_t) = 0$. The processes u_t and x_t may be serially correlated and heteroskedastic, but are assumed to satisfy the following high-level assumption.

A1. (i) $T^{-1/2}\sum_{t=1}^{\lfloor T\cdot\rfloor} x_t u_t \to_d \Omega^{1/2} W_k(\cdot)$, where Ω is positive definite, $W_k(\cdot)$ is a k-dimensional Wiener process and $\lfloor\cdot\rfloor$ denotes the integer part of the argument.
(ii) $\sup_{0\le r\le 1}\left|T^{-1}\sum_{t=1}^{\lfloor Tr\rfloor} x_t x_t' - rQ\right| \to_p 0$, where Q is positive definite.

Assumption A1 is due to KVB and is discussed there. Suffice it to say that this assumption is satisfied under a wide range of primitive moment and memory conditions on u_t and x_t. An important implication of A1 is that the OLS estimator $\hat\beta = \left(\sum_{t=1}^T x_t x_t'\right)^{-1}\left(\sum_{t=1}^T x_t y_t\right)$ is root-T consistent and asymptotically normal:

$$\sqrt{T}\left(\hat\beta - \beta\right) \to_d \mathcal{N}\left(0, Q^{-1}\Omega Q^{-1}\right).$$

Suppose the objective is to test hypotheses on β, treating the serial correlation properties of u_t as a nonparametric nuisance feature. Specifically, suppose the hypothesis to be tested $H_0 : R\beta = r$, where R is a $q\times k$ matrix (of rank q) and r is a q-vector.

The standard approach is to base inference on a test statistic of the form

$$F_{HAC} = T\left(R\hat\beta - r\right)'\left(R\hat Q^{-1}\hat\Omega_{HAC}\hat Q^{-1}R'\right)^{-1}\left(R\hat\beta - r\right),$$

where $\hat Q = T^{-1}\sum_{t=1}^T x_t x_t'$ is a consistent estimator of Q (Assumption A1 (ii)), while $\hat\Omega_{HAC}$ is a consistent estimator of Ω, the long-run covariance matrix of $x_t u_t$. Although consistent estimators of Ω are available under conditions resembling A1 (e.g. Andrews (1991), Andrews and Monahan (1992), Hansen (1992), de Jong and Davidson (2000), Newey and West (1987, 1994) and Robinson (1991)), the size properties of tests based on F_{HAC} can be quite unsatisfactory in finite samples (e.g. den Haan and Levin (1997)).

To the extent that the poor size properties of F_{HAC} are likely to be due to the fact that distributional approximations based on conventional asymptotic theory fail to capture the finite-sample variability in $\hat{\Omega}_{HAC}$, it seems plausible that tests with better size properties can be obtained by employing an "estimator" of Ω whose limiting distribution is non-degenerate. Corroboration of this conjecture has been provided by KVB, which proposes the test statistic

$$F_{KVB} = T\left(R\hat{\beta} - r\right)'\left(R\hat{Q}^{-1}\hat{\Omega}_{KVB}\hat{Q}^{-1}R'\right)^{-1}\left(R\hat{\beta} - r\right),$$

where

$$\hat{\Omega}_{KVB} = T^{-2}\sum_{t=1}^{T-1}\hat{S}_t\hat{S}_t', \qquad \hat{S}_t = \sum_{s=1}^{t} x_s\left(y_s - x_s'\hat{\beta}\right).$$

Unlike $\hat{\Omega}_{HAC}$, $\hat{\Omega}_{KVB}$ is not a consistent estimator of Ω. Nonetheless, F_{KVB} is asymptotically pivotal under H_0 and the associated test has good finite-sample size properties and respectable (finite-sample and local asymptotic) power properties. Recently, Kiefer and Vogelsang (2002a) have shown that $\hat{\Omega}_{KVB}$ equals one half times the kernel estimator of Ω computed using the Bartlett kernel with the bandwidth parameter equal to the sample size, while Kiefer and Vogelsang (2002b) have shown that the Bartlett kernel dominates other popular kernels in terms of the local asymptotic power of tests based on kernel estimators implemented with the bandwidth parameter equal to the sample size.

The present paper studies test statistics of the form

$$F_{\kappa} = T\left(R\hat{\beta} - r\right)'\left(R\hat{Q}^{-1}\hat{\Omega}_{\kappa}\hat{Q}^{-1}R'\right)^{-1}\left(R\hat{\beta} - r\right),$$

where

$$\hat{\Omega}_{\kappa} = T^{-2}\sum_{t=1}^{T-1}\kappa\left(\frac{t}{T}\right)\hat{S}_t\hat{S}_t'$$

and $\kappa(\cdot)$ is a weight function satisfying the following assumption.

A2. $\kappa : (0,1) \to [0,\infty)$ is Lipschitz continuous; that is, there exists a finite constant M_{κ} such that $|\kappa(r) - \kappa(s)| \leq M_{\kappa}|r - s|$ for all $0 \leq r \leq s \leq 1$.

When the weight function $\kappa(\cdot)$ is constant, the statistic F_κ is equivalent to F_{KVB}. On the other hand, nonconstant weight functions give rise to test statistics that are not covered by the results of Kiefer and Vogelsang (2002b). The statistic F_κ generalizes F_{KVB} in a manner analogous to Anderson and Darling's (1952) generalization of the Cramér-von Mises goodness of fit test. Specifically, the limiting distribution of $\hat{\Omega}_{KVB}$ is of the Cramér-von Mises variety, whereas the limiting representation of $\hat{\Omega}_\kappa$ turns out to be a multivariate version of the statistic W^2 appearing in equation (4.5) of Anderson and Darling (1952).

3. SIZE PROPERTIES IN A GAUSSIAN LOCATION MODEL

In Monte Carlo experiments, KVB and Bunzel, Kiefer, and Vogelsang (2001) have found the size properties of F_{KVB} to be superior to those of F_{HAC}. A heuristic explanation of these findings can be found in Bunzel, Kiefer, and Vogelsang (2001, p. 1093), but to the best of this author's knowledge no previous paper has attempted to use higher-order asymptotic theory to provide an analytical explanation of the encouraging size performance of the KVB procedure. As a first step in that direction, this section derives the rate of convergence of F_κ to its (non-normal) limiting null distribution under the assumption that y_t is generated by the Gaussian location model

$$y_t = \beta + u_t, \qquad t = 1, \ldots, T, \tag{2}$$

where

A1*. $u_t = \psi(L)\eta_t$, where $\eta_t \sim i.i.d.\ \mathcal{N}(0,1)$ and $\psi(L) = \sum_{i=0}^{\infty} \psi_i L^i$ is a lag polynomial with $\psi(1) \neq 0$ and $\sum_{i=1}^{\infty} i|\psi_i| < \infty$.

The location model is the simplest possible special case of the general regression model presented in Section 2, but is nonetheless of some empirical interest (e.g. Diebold and Mariano (1995), Zambrano and Vogelsang (2000)). Employing a model similar to the one studied here, Velasco and Robinson (2001) derive Edgeworth expansions of the distribution of F_{HAC} under the assumption that $\hat{\Omega}_{HAC}$ belongs to a certain class of kernel estimators. The leading term in the asymptotic expansion of the distribution function of F_{HAC} is no smaller than $O\left(T^{-1/2}\right)$ when the bandwidth expansion rate is such that the order of the asymptotic mean squared error (MSE) of $\hat{\Omega}_{HAC}$ is minimized (Velasco and Robinson (2001, Section 4)). In contrast, Theorem 1 shows that even if inference is based on first-order asymptotic theory, the ERP is $O(T^{-1})$ when the hypothesis $H_0 : \beta = \beta_0$ is tested by means of the test statistic introduced in this paper, viz.

$$\frac{T\left(\hat{\beta}-\beta_0\right)^2}{T^{-2}\sum_{t=1}^{T-1}\kappa\left(t/T\right)\hat{S}_t^2}, \qquad \hat{S}_t=\sum_{s=1}^{t}\left(y_s-\hat{\beta}\right),$$

where $\hat{\beta}=T^{-1}\sum_{t=1}^{T}y_t$ is the sample mean.

Theorem 1. *Suppose y_t is generated by (2) and suppose assumptions A1* and A2 hold. Then*

$$\sup_{c\in\mathbb{R}}\left|\Pr\left(\frac{T\left(\hat{\beta}-\beta\right)^2}{T^{-2}\sum_{t=1}^{T-1}\kappa\left(t/T\right)\hat{S}_t^2}\le c\right)-\Pr\left(\frac{\mathcal{Z}^2}{\int_0^1\kappa\left(r\right)B\left(r\right)^2dr}\le c\right)\right|=O\left(T^{-1}\right),$$

where $B(\cdot)$ is a Brownian bridge and $\mathcal{Z}\sim\mathcal{N}(0,1)$ is independent of $B(\cdot)$.

The proof of Theorem 1 establishes a moment bound on the remainder term in a stochastic expansion of the test statistic and then translates the moment bound into a bound on the ERP by applying the following remarkable implication of Stein's lemma.

Lemma 2 [Shorack (2000)]. *For any random variables ξ and Δ,*

$$\sup_{c\in\mathbb{R}}\left|\Pr\left(\xi+\Delta\le c\right)-\Pr\left(\xi\le c\right)\right|\le 4\left(1+\sqrt{E\left[\xi^2\right]}\right)\sqrt{E\left[\Delta^2\right]}.$$

Normality plays an important simplifying role in the proof of Theorem 1, greatly facilitating the construction of a good coupling between the test statistic and its (non-normal) asymptotic representation. It is plausible that an extension of Theorem 1 to non-Gaussian time series can be based on Götze and Tikhomirov (2001), but an investigation along those lines is beyond the scope of this paper.

In view of Theorem 1, the fact that F_{KVB} dominates F_{HAC} in terms of finite-sample size properties is consistent with the predictions of higher-order asymptotic theory. Theorem 1 therefore complements the Monte Carlo results of KVB and Bunzel, Kiefer, and Vogelsang (2001) and sheds new light on these. Heuristically, the fast rate of decay of the ERP of F_κ is achieved by employing a standardization factor $\hat{\Omega}_\kappa$ whose finite-sample distribution is well approximated by its asymptotic counterpart. Indeed, the contribution of stochastic difference between $\hat{\Omega}_\kappa$ and its limiting representation to the asymptotic expansion reported in Theorem 1 is of the same order of magnitude as the contribution due to the stochastic difference between $T^{1/2}\left(\hat{\beta}-\beta\right)$ and its limiting normal distribution, whereas only the discrepancy between $\hat{\Omega}_{HAC}$ and its probability limit is reflected in the leading term in the asymptotic expansion of the ERP of F_{HAC} (Velasco and Robinson (2001, Section 4)).

Under the (admittedly restrictive) assumptions of Theorem 1, the ERP of F_κ compares favorably with the ERP of existing bootstrap-based procedures. The best currently available results on the performance of bootstrap-based autocorrelation robust inference procedures in the presence of nonparametric autocorrelation would appear to be those of Choi and Hall (2000), Götze and Künsch (1996) and Inoue and Shintani (2001).[1] Choi and Hall (2000) state conditions under which the sieve bootstrap delivers an ERP with a polynomial rate of decay arbitrarily close to $O(T^{-1})$, while Götze and Künsch (1996) and Inoue and Shintani (2001) show that the ERP is no better than $O\left(T^{-2/3}\right)$ when the block bootstrap is applied to F_{HAC} and $\hat{\Omega}_{HAC}$ is constructed using a kernel guaranteed to yield positive semidefinite estimates. It would be of interest to know whether the bootstrap can be successfully applied to F_κ.

Remarks. (i) One-sided tests of $H_0 : \beta = \beta_0$ can be based on the statistic

$$\frac{T^{1/2}\left(\hat{\beta} - \beta_0\right)}{\sqrt{T^{-2}\sum_{t=1}^{T-1}\kappa(t/T)\hat{S}_t^2}}.$$

The null distribution of this statistic is symmetric under the assumptions of Theorem 1. As a consequence,

$$\Pr\left(\frac{T^{1/2}\left(\hat{\beta} - \beta\right)}{\sqrt{T^{-2}\sum_{t=1}^{T-1}\kappa(t/T)\hat{S}_t^2}} \leq c\right) = \frac{1}{2} + \frac{1}{2}\Pr\left(\frac{T\left(\hat{\beta} - \beta\right)}{T^{-2}\sum_{t=1}^{T-1}\kappa(t/T)\hat{S}_t^2} \leq c^2\right)$$

for any $c > 0$ (with an analogous result holding for $c < 0$) and it follows that the ERP of one-sided tests is $O(T^{-1})$ under the assumptions of Theorem 1.

(ii) Lemma 2 may be of independent interest. In particular, Lemma 2 is a useful complement to the following result, often attributed to Chibisov (1972) or Sargan and Mikhail (1971): for any random variables ξ and Δ,

$$\sup_{c\in\mathbb{R}}|\Pr(\xi + \Delta \leq c) - \Pr(\xi \leq c)| \leq \inf_{\zeta>0}\left(\Pr(|\Delta| > \zeta) + \sup_{c\in\mathbb{R}}\Pr(|\xi - c| < \zeta)\right);$$

see Rothenberg (1984) and Taniguchi and Kakizawa (2000). It is difficult to see how a proof of Theorem 1 can be based on this inequality (as opposed to Lemma 2).

[1] Contemporary reviews of bootstrap methods for time series can be found in Bühlmann (2002), Härdle, Horowitz, and Kreiss (2001) and Politis (2002).

4. Local Asymptotic Power

Theorem 1 of the previous section shows that the favorable higher-order size properties of F_{KVB} are shared by F_κ irrespective of the choice of the weight function $\kappa(\cdot)$. It therefore appears sensible to let the choice of $\kappa(\cdot)$ be based on power considerations. This section addresses the local asymptotic power properties of the test based on F_κ in the context of the general regression model of Section 2.

As do conventional testing procedures, the test based on F_κ has nontrivial power against alternatives of the form $R\beta - r = O\left(T^{-1/2}\right)$. A precise statement is provided in Theorem 3, which characterizes the limiting distribution of F_κ under local alternatives of the form

$$H_1 : R\beta = r + T^{-1/2}\left(RQ^{-1}\Omega Q^{-1}R'\right)^{1/2}\delta,$$

where δ is a q-vector of constants.

Theorem 3. *Suppose y_t is generated by* (1) *and suppose assumptions A1-A2 hold. If $\delta = T^{1/2}\left(RQ^{-1}\Omega Q^{-1}R'\right)^{-1/2}(R\beta - r)$ is fixed as T increases without bound, then*

$$F_\kappa \to_d \left(\mathcal{Z}_q + \delta\right)'\left(\int_0^1 \kappa(r)\, B_q(r)\, B_q(r)'\, dr\right)^{-1}\left(\mathcal{Z}_q + \delta\right),$$

where $B_q(\cdot)$ is a q-dimensional Brownian bridge and $\mathcal{Z}_q \sim \mathcal{N}(0, I_q)$ is independent of $B_q(\cdot)$.

Under H_0, F_κ is asymptotically pivotal with a (nonstandard) limiting distribution that depends on the weight function $\kappa(\cdot)$ and q, the number of restrictions. More generally, the limiting behavior under local alternatives depends on $\kappa(\cdot)$, q and the noncentrality parameter $\delta'\delta$.

In the search for a weight function such that the test based on F_κ enjoys good power properties, it turns out to be fruitful to study the related problem of choosing $\kappa(\cdot)$ in such a way that $\hat{\Omega}_\kappa$ enjoys good properties when viewed as an estimator of Ω. As is well known, a variance estimator with optimal MSE properties does not necessarily deliver a test statistic with good size and/or power properties (e.g. Andrews (1991, p. 828), Simonoff (1993)). A seemingly pathological exception to that rule occurs when the variance estimators carry no information about the parameter of interest and a "perfect" variance estimator (having zero MSE) exists. In that case, an optimal (from an MSE point of view) variance estimator does indeed give rise to a test with optimal power properties. Somewhat surprisingly, perhaps, it turns out that $\hat{\Omega}_\kappa$ is nearly "perfect" if the weight function $\kappa(\cdot)$ is chosen appropriately. Indeed, it is shown in Appendix A that while the asymptotic truncated MSE of $\hat{\Omega}_\kappa$

is strictly positive for any weight function $\kappa(\cdot)$, the asymptotic truncated MSE can be made arbitrarily small by employing $\kappa_\varepsilon(\cdot)$ with ε close to zero, where

$$\kappa_\varepsilon(r) = \frac{1}{\frac{3-2\varepsilon}{3(1-\varepsilon)^2} + 2\log\left(\frac{1-\varepsilon}{\varepsilon}\right)} \cdot \min\left(r^{-2}(1-r)^{-2}, \varepsilon^{-2}(1-\varepsilon)^{-2}\right) \tag{3}$$

for any $\varepsilon \in (0, 1/2]$ and any $r \in (0,1)$. As a consequence, the local asymptotic power function of a test based on F_κ can be made arbitrarily close to the asymptotic power envelope[2] by employing $\kappa_\varepsilon(\cdot)$ with ε sufficiently close to zero.

Admittedly, the finite sample relevance of this near-optimality result is limited by the fact that the F_κ-statistics based on $\kappa_\varepsilon(\cdot)$ and $\kappa_{\varepsilon'}(\cdot)$ are equivalent whenever $\max(\varepsilon, \varepsilon') < T^{-1}$. Nevertheless, it seems reasonable to believe that small values of the truncation parameter ε deliver test statistics F_{κ_ε} with good power properties. In particular, F_{κ_ε} should dominate $F_{KVB} = F_{\kappa_{0.5}}$ even for moderately small values of ε. A verification of that conjecture is provided by Figure 1, which plots the local asymptotic power of $F_{\kappa_{0.01}}$ and F_{KVB} along with the power envelope in the case where a single restriction is being tested.[3]

FIGURE 1 ABOUT HERE

Although its power curve lies uniformly below the power envelope, the test based on $F_{\kappa_{0.01}}$ dominates F_{KVB} in terms of local asymptotic power. Local asymptotic power results for one-sided tests (described in remark (i) below) and for tests of multiple hypotheses are qualitatively similar and are omitted to conserve space. Also omitted are results for alternative weight functions such as $\kappa_{0.005}$ and $\kappa_{0.02}$, as these are quite similar to those for $\kappa_{0.01}$. Specifically, the local asymptotic power of the test based $F_{\kappa_{0.005}}$ is slightly higher than that of the test based on $F_{\kappa_{0.01}}$, which in turn is slightly superior to the test based on $F_{\kappa_{0.02}}$. In short, the statistic $F_{\kappa_{0.01}}$ seems to be an attractive alternative to F_{KVB}. Table 1 gives selected percentiles of the limiting null distribution of $F_{\kappa_{0.01}}$.

TABLE 1 ABOUT HERE

[2] The asymptotic power envelope is the local asymptotic power function corresponding to the (infeasible) test based on $T\left(R\hat{\beta} - r\right)'\left(RQ^{-1}\Omega Q^{-1}R'\right)^{-1}\left(R\hat{\beta} - r\right)$, whose limiting distribution is the noncentral χ^2 distribution with q degrees of freedom and noncentrality parameter $\delta'\delta$ under the assumption of Theorem 3. The test based on F_{HAC} attains the asymptotic power envelope whenever $\hat{\Omega}_{HAC}$ is a consistent estimator of Ω.

[3] The curves were generated by taking $50,000$ draws from the distribution of the discrete approximation (based on $1,000$ steps) to the limiting random variables.

Remarks. (i) One-sided tests of a single restriction can be based on

$$t_\kappa = \frac{T^{1/2}\left(R\hat{\beta} - r\right)}{\sqrt{R\hat{Q}^{-1}\hat{\Omega}_\kappa\hat{Q}^{-1}R'}}.$$

Under the assumptions of Theorem 3,

$$t_\kappa \to_d \frac{\mathcal{Z}_1 + \delta}{\sqrt{\int_0^1 \kappa(r) B_1(r)^2 dr}}.$$

Now, $t_\kappa^2 = F_\kappa$ and the limiting null distribution of t_κ is symmetric, so the percentiles of the limiting null distribution are related to those of the limiting null distribution of F_κ in an obvious way: Under H_0, $\Pr(t_\kappa \le c) = \frac{1}{2} + \frac{1}{2}\Pr(F_\kappa \le c^2)$ for any $c > 0$, with an analogous result holding for $c < 0$.

(ii) The scale factor $1/\left(\frac{3-2\varepsilon}{3(1-\varepsilon)^2} + 2\log\left(\frac{1-\varepsilon}{\varepsilon}\right)\right)$ appearing in the definition of $\kappa_\varepsilon(\cdot)$ is redundant for testing purposes, but is included in order to facilitate comparison of the percentiles of the limiting null distribution of F_{κ_ε} with the percentiles of the corresponding $\chi^2(q)$ distribution. Specifically, the scale normalization implies that

$$\mathcal{Z}_q'\left(\int_0^1 \kappa_\varepsilon(r) B_q(r) B_q(r)' dr\right)^{-1} \mathcal{Z}_q \to_d \chi^2(q)$$

as $\varepsilon \downarrow 0$ because $\int_0^1 \kappa_\varepsilon(r) B_q(r) B_q(r)' dr \to_p I_k$ as $\varepsilon \downarrow 0$ (cf. Appendix A).

(iii) In cases where the series $x_t u_t$ is persistent, a test statistic based on a prewhitened version of $\hat{\Omega}_\kappa$ might be expected to outperform F_κ in terms of small sample size properties. By analogy with Andrews and Monahan (1992), a VAR(1) prewhitened version of $\hat{\Omega}_\kappa$ can be constructed as follows. Let

$$\hat{\Omega}_\kappa^{PW} = \left(I_k - \hat{A}\right)^{-1}\left(T^{-2}\sum_{t=1}^{T-1}\kappa\left(\frac{t}{T}\right)\tilde{S}_t\tilde{S}_t'\right)\left(I_k - \hat{A}'\right)^{-1},$$

where $\hat{A}$ is a $k \times k$ matrix and $\tilde{S}_t = \hat{S}_t - \hat{A}\hat{S}_{t-1}$ for $t = 2, \ldots, T$. A convenient choice of $\hat{A}$ is the least squares estimator $\hat{A}_{LS} = \left(\sum_{t=2}^T \hat{v}_t\hat{v}_{t-1}'\right)\left(\sum_{t=2}^T \hat{v}_{t-1}\hat{v}_{t-1}'\right)^{-1}$, where $\hat{v}_t = x_t\left(y_t - x_t'\hat{\beta}\right)$. Under weak regularity conditions, $\hat{A}_{LS}$ meets the requirements (on $\hat{A}$) of the following corollary to Theorem 3.

Corollary 4. *If the assumptions of Theorem 3 hold and* $\hat{A} - A = o_p(1)$ *for some* A *such that* $I_k - A$ *is nonsingular, then*

$$F_\kappa^{PW} = T\left(R\hat{\beta} - r\right)'\left(R\hat{Q}^{-1}\hat{\Omega}_\kappa^{PW}\hat{Q}^{-1}R'\right)^{-1}\left(R\hat{\beta} - r\right)$$

$$\to_d \left(\mathcal{Z}_q + \delta\right)'\left(\int_0^1 \kappa(r) B_q(r) B_q(r)' dr\right)^{-1}\left(\mathcal{Z}_q + \delta\right).$$

(iv) Theorem 3 generalizes in an obvious way to tests of nonlinear hypotheses. Moreover, proceeding as in Vogelsang (2002), it is straightforward to generalize the first-order asymptotic theory of this section to models estimated by the generalized method of moments.

5. Finite Sample Evidence

A small Monte Carlo experiment is conducted in order to explore the extent to which the predictions from the asymptotic theory presented in the previous section are likely to be borne out in finite samples. For brevity, only a location model is considered. Samples of size $T = 100$ are generated according to the following model:

$$y_t = \beta + u_t, \qquad t = 1, \ldots, T,$$

where

$$u_t = \rho u_{t-1} + e_t, \qquad t = 2, \ldots, T,$$

$u_1 \sim \mathcal{N}(0, (1-\rho)/(1+\rho))$ and $e_t \sim i.i.d.\ \mathcal{N}\left(0, (1-\rho)^2\right)$ with $\{e_t : t \geq 2\}$ independent of u_1. By construction, the error u_t follows a stationary AR(1) process with unit long-run variance for all values of ρ. The null and alternative hypotheses are $H_0 : \beta = 0$ and $H_1 : \beta \neq 0$, respectively. The parameter of interest, β, takes on value in the set $\{0, 0.1, 0.2, 0.3, 0.4\}$, while the nuisance parameter ρ takes on values in the set $\{-0.5, -0.3, 0, 0.3, 0.5, 0.7, 0.9, 0.95\}$. The quality of the distributional approximation A1(i) is known to be quite reasonable unless $|\rho|$ is close to one. The values $\rho = 0.90$ and $\rho = 0.95$ are included in order to compare various autocorrelation robust inference procedures in terms of their ability to accommodate nearly integrated errors without employing alternative distributional approximations (such as those of Chan and Wei (1987) and Phillips (1987, 1988)).

Six test statistics are considered. The first statistic is F_{HAC}, where $\hat{\Omega}_{HAC}$ is a kernel estimator of Ω implemented with the quadratic spectral kernel along with a plug-in bandwidth (see Andrews (1991) for details). The second statistic, denoted F_{HAC}^{PW}, uses Andrews and Monahan's (1992) AR(1) prewhitened version of $\hat{\Omega}_{HAC}$. The third and fourth statistics are F_{KVB} and F_{KVB}^{PW}, respectively, where F_{KVB}^{PW} is computed using AR(1) prewhitened version of $\hat{\Omega}_{KVB}$ (see Bunzel, Kiefer, and Vogelsang (2001) for details). Finally, the fifth and sixth statistics are F_κ and F_κ^{PW}, respectively, where $\kappa(\cdot) = \kappa_{0.01}(\cdot)$. Table 2 summarizes the results. For each test statistic, the row corresponding to $\beta = 0$ reports observed rejection rates (based on 5,000 Monte Carlo replications) of 5% tests using asymptotic critical values, while the rows corresponding to $\beta \neq 0$ report size-adjusted power.

TABLE 2 ABOUT HERE

The prewhitened versions of the three test statistics have significantly better size properties than the test statistics that do not employ prewhitening and the improvement in terms of size distortions does not come at the expense of size-adjusted power. Among the prewhitened test statistics, F_{KVB}^{PW} and F_κ^{PW} have fairly similar size properties and both statistics are superior to F_{HAC}^{PW} in terms of size. For moderate values of ρ, the ranking of the test statistics in terms of size-adjusted power is consistent with the predictions from asymptotic theory. Indeed, as suggested by Figure 1, F_{HAC}^{PW} is more powerful than F_κ^{PW}, which in turn dominates F_{KVB}^{PW} in terms of size-adjusted power. Unsurprisingly, the numerical results are somewhat different when u_t is highly persistent. The difference between the size of the tests becomes increasingly pronounced as ρ approaches unity, with F_κ^{PW} doing somewhat better than F_{KVB}^{PW} and much better than F_{HAC}^{PW}. Interestingly, the ranking of F_{KVB}^{PW} and F_κ^{PW} in terms of size-adjusted power is reversed for $\rho \in \{0.9, 0.95\}$.

Overall, the Monte Carlo results can be summarized as follows. On the one hand, the results are favorable to the test statistic advocated in this paper, the F_κ^{PW} statistic implemented using $\kappa_{0.01}$, as that statistic dominates F_{HAC}^{PW} and F_{KVB}^{PW} in terms of size and power, respectively. In this sense, the procedure proposed herein is arguably superior to existing procedures. On the other hand, the fact that conventional test statistics dominate in terms of size-adjusted power indicates that an even better procedure may be available if improved approximations to the null distribution of F_{HAC}^{PW} and/or F_{HAC} can be found.[4]

[4] It is conceivable that this may be achieved by means of an approach resembling that advocated by Bekker (1994) and Donald and Newey (2001) in the context of instrumental variables regression with many instruments. A recent paper along these lines, Kiefer and Vogelsang (2002c), finds that more accurate distributional approximations can be obtained by modeling the bandwidth (employed in the construction of $\hat{\Omega}_{HAC}$) as a fixed proportion of the sample size when developing first-order asymptotic theory for F_{HAC}.

6. Appendix A: Asymptotic Properties of $\hat{\Omega}_\kappa$

By analogy with Andrews (1991), a comparison of different weight functions can be based on the asymptotic truncated moments of $\hat{\Omega}_\kappa$. For any $h > 0$, let

$$\hat{\Omega}_{\kappa,h} = \min\left(\max\left(\hat{\Omega}_\kappa, -h\iota_k\iota_k'\right), h\iota_k\iota_k'\right),$$

where ι_k is a k-vector of ones and min (max) is defined element-by-element in the obvious way.

Lemma 5. *If the assumptions of Theorem 3 hold, then*

$$\lim_{h\to\infty}\lim_{T\to\infty} E\left[\hat{\Omega}_{\kappa,h}\right] = \Omega \int_0^1 \kappa(r)\, m(r)\, dr,$$

and

$$\lim_{h\to\infty}\lim_{T\to\infty} Var\left[vec\left(\hat{\Omega}_{\kappa,h}\right)\right] = \left(I_{k^2} + K_{k^2}\right)(\Omega\otimes\Omega)\int_0^1\int_0^1 \kappa(r)\, C(r,s)\, \kappa(s)\, drds,$$

where $m(r) = r(1-r)$, $C(r,s) = \min(r,s)^2(1-\max(r,s))^2$ *and* K_{k^2} *is the* $k^2\times k^2$ *commutation matrix.*

Any scale transformation of $\kappa(\cdot)$ preserves the local asymptotic power properties of F_κ. In view of Lemma 5, it seems natural to normalize the scale of $\kappa(\cdot)$ by imposing the "unbiasedness" condition $\int_0^1 \kappa(r)\, m(r)\, dr = 1$. Subject to this normalization, one might attempt to find a weight function that minimizes the asymptotic truncated variance of $\hat{\Omega}_\kappa$, viz.

$$\left(I_{k^2} + K_{k^2}\right)(\Omega\otimes\Omega)\int_0^1\int_0^1 \kappa(r)\, C(r,s)\, \kappa(s)\, drds.$$

As it turns out, this minimization problem does not have a solution. On the one hand, $C(\cdot,\cdot)$ is positive definite in the sense that $\int_0^1\int_0^1 f(r)\, C(r,s)\, f(s)\, drds > 0$ for any square integrable function $f(\cdot)$. On the other hand, for any $\varepsilon\in(0,1/2]$, let $\kappa_\varepsilon(\cdot)$ be defined as in equation (3). By construction, $\int_0^1 \kappa_\varepsilon(r)\, m(r)\, dr = 1$. Moreover,

$$\begin{aligned}\int_0^1\int_0^1 \kappa_\varepsilon(r)\, C(r,s)\, \kappa_\varepsilon(s)\, drds &= \frac{\frac{150-408\varepsilon+280\varepsilon^2}{45(1-\varepsilon)^4} - \frac{4-8\varepsilon}{(1-\varepsilon)^2} + 8\log\left(\frac{1-\varepsilon}{\varepsilon}\right)}{\left(\frac{3-2\varepsilon}{3(1-\varepsilon)^2} + 2\log\left(\frac{1-\varepsilon}{\varepsilon}\right)\right)^2} \\ &= O\left(\frac{1}{\log\left(\varepsilon^{-1}\right)}\right)\end{aligned}$$

as $\varepsilon \downarrow 0$. As a consequence, $\lim_{h\to\infty}\lim_{T\to\infty} Var\left[vec\left(\hat{\Omega}_{\kappa,h}\right)\right]$ can be made arbitrarily close to zero even if $\lim_{h\to\infty}\lim_{T\to\infty} E\left[\hat{\Omega}_{\kappa,h}\right] = \Omega$ is imposed and no optimal (in the sense of minimal asymptotic truncated MSE) estimator exists among estimators of the form $\hat{\Omega}_{\kappa}$.

Remark. The weight function $\kappa_{\varepsilon}(\cdot)$ is a truncated and rescaled version of $\kappa^{*}(\cdot)$, where $\kappa^{*}(r) = r^{-2}(1-r)^{-2}$ for any $r \in (0,1)$. Truncation is introduced because $\int_0^1 \kappa^{*}(r)\, m(r)\, dr = \infty$, while the rescaling achieves $\int_0^1 \kappa_{\varepsilon}(r)\, m(r)\, dr = 1$.

The functional form of $\kappa^{*}(\cdot)$ is suggested by the following observation. A variational argument can be used to show that $\kappa(\cdot)$ minimizes

$$\int_0^1 \int_0^1 \kappa(r)\, C(r,s)\, \kappa(s)\, dr ds$$

subject to $\int_0^1 \kappa(r)\, m(r)\, dr = 1$ only if $\int_0^1 C(\cdot,s)\, \kappa(s)\, ds$ is proportional to $m(\cdot)$. In turn, it is not hard to show that $\int_0^1 C(\cdot,s)\, \kappa(s)\, ds$ is proportional to $m(\cdot)$ only if $\kappa(\cdot)$ is proportional to $\kappa^{*}(\cdot)$.

7. Appendix B: Proofs

Proof of Theorem 1. Let $X = (X_N, X_D')'$, where $X_N = \psi(1)^{-1} T^{1/2}\left(\hat{\beta}-\beta\right)$ and $X_D = \psi(1)^{-1} T^{-1}\left(\sqrt{\kappa(1/T)}\hat{S}_1, \ldots, \sqrt{\kappa((T-1)/T)}\hat{S}_{T-1}\right)'$. Under the assumptions of Theorem 1, $X =_d \Sigma^{1/2}\eta$, where $\eta \sim \mathcal{N}(0, I_T)$,

$$\Sigma = \begin{pmatrix} \sigma_{NN} & \sigma_{DN}' \\ \sigma_{DN} & \Sigma_{DD} \end{pmatrix}$$

is the covariance matrix of X (partitioned in the obvious way) and "$=_d$" signifies equality in distribution. For any $c \in \mathbb{R}$,

$$\Pr\left(\frac{T\left(\hat{\beta}-\beta\right)^2}{T^{-2}\sum_{t=1}^{T-1}\kappa(t/T)\hat{S}_t^2} \le c\right) = \Pr\left(X'\Psi_c X \le 0\right) = \Pr\left(\eta'\Upsilon_c\eta \le 0\right),$$

where

$$\Psi_c = \begin{pmatrix} \min(1, c^{-1}) & 0 \\ 0 & -\min(1, c)\, I_{T-1} \end{pmatrix}$$

and

$$\begin{aligned} \Upsilon_c &= \Sigma^{1/2\prime}\Psi_c\Sigma^{1/2} \\ &= \begin{pmatrix} \min(1, c^{-1})\,\sigma_{NN} - \min(1, c)\,\sigma_{NN}^{-1}\sigma_{DN}'\sigma_{DN} & -\min(1, c)\,\sigma_{NN}^{-1/2}\sigma_{DN}'\Sigma_{DD.N}^{1/2} \\ -\min(1, c)\,\sigma_{NN}^{-1/2}\Sigma_{DD.N}^{1/2\prime}\sigma_{DN} & -\min(1, c)\,\Sigma_{DD.N} \end{pmatrix}, \end{aligned}$$

where $\Sigma_{DD.N} = \Sigma_{DD} - \sigma_{NN}^{-1}\sigma_{DN}\sigma_{DN}'$. Similarly,

$$\Pr\left(\frac{\mathcal{Z}^2}{\int_0^1 \kappa(r)\, B(r)^2\, dr} \le c\right) = \Pr\left(\min\left(1, c^{-1}\right)\mathcal{Z}^2 - \min(1, c)\int_0^1 \kappa(r)\, B(r)^2\, dr \le 0\right).$$

Let $\kappa_T(\cdot) = \kappa(\lfloor T\cdot\rfloor / T)$ and $B_T(\cdot) = B(\lfloor T\cdot\rfloor / T)$ denote discretized versions of $\kappa(\cdot)$ and $B(\cdot)$, respectively. It is easy to show that

$$\min\left(1, c^{-1}\right)\mathcal{Z}^2 - \min(1, c)\int_0^1 \kappa_T(r)\, B_T(r)^2\, dr =_d \eta'\dot{\Upsilon}_c\eta, \tag{4}$$

where

$$\dot{\Upsilon}_c = \begin{pmatrix} \min\left(1, c^{-1}\right) & 0 \\ 0 & -\min\left(1, c\right) \dot{\Sigma}_{DD} \end{pmatrix}$$

and $\dot{\Sigma}_{DD}$ is the $(T-1) \times (T-1)$ matrix whose (i, j)th element is

$$\dot{\Sigma}_{DD}\left(i, j\right) = T^{-2} \sqrt{\kappa\left(i/T\right)} \sqrt{\kappa\left(j/T\right)} \left(\min\left(i, j\right) - ij/T\right).$$

Using the distributional equality (4),

$$\Pr\left(\frac{T\left(\hat{\beta} - \beta\right)^2}{T^{-2} \sum_{t=1}^{T-1} \kappa\left(t/T\right) \hat{S}_t^2} \leq c\right) - \Pr\left(\frac{\mathcal{Z}^2}{\int_0^1 \kappa\left(r\right) B\left(r\right)^2 dr} \leq c\right) = \Xi_1\left(c\right) + \Xi_2\left(c\right),$$

where

$$\Xi_1\left(c\right) = \Pr\left(\eta' \Upsilon_c \eta \leq 0\right) - \Pr\left(\eta' \dot{\Upsilon}_c \eta \leq 0\right)$$

and

$$\begin{aligned} \Xi_2\left(c\right) &= \Pr\left(\min\left(1, c^{-1}\right) \mathcal{Z}^2 - \min\left(1, c\right) \int_0^1 \kappa_T\left(r\right) B_T\left(r\right)^2 dr \leq 0\right) \\ &\quad - \Pr\left(\min\left(1, c^{-1}\right) \mathcal{Z}^2 - \min\left(1, c\right) \int_0^1 \kappa\left(r\right) B\left(r\right)^2 dr \leq 0\right). \end{aligned}$$

The proof of Theorem 1 will be completed by showing that $\sup_{c \in \mathbb{R}} \left|\Xi_1\left(c\right)\right| = O\left(T^{-1}\right)$ and $\sup_{c \in \mathbb{R}} \left|\Xi_2\left(c\right)\right| = O\left(T^{-1}\right)$.

Now, using the properties of the normal distribution (Magnus and Neudecker (1988, Theorem 12.12)) and simple algebra,

$$E\left[\left(\eta' \dot{\Upsilon}_c \eta\right)^2\right] = \left[tr\left(\dot{\Upsilon}_c\right)\right]^2 + 2tr\left(\dot{\Upsilon}_c^2\right) \leq 3\left(1 + \|\kappa\|\right)^2,$$

where $\|\kappa\| = \sup_{0<r<1} \left|\kappa\left(r\right)\right| < \infty$ under A2. Using this display and Lemma 2,

$$\sup_{c\in\mathbb{R}}|\Xi_1(c)| \leq 12(1+\|\kappa\|)\sqrt{E\left[\left(\eta'\Upsilon_c\eta - \eta'\dot{\Upsilon}_c\eta\right)^2\right]}$$

and

$$\sup_{c\in\mathbb{R}}|\Xi_2(c)| \leq 12(1+\|\kappa\|)\sqrt{E\left[\left(\int_0^1 \kappa_T(r)\,B_T(r)^2\,dr - \int_0^1 \kappa(r)\,B(r)^2\,dr\right)^2\right]}.$$

It therefore suffices to establish the following bounds:

$$E\left[\left(\eta'\Upsilon_c\eta - \eta'\dot{\Upsilon}_c\eta\right)^2\right] = O\left(T^{-2}\right), \tag{5}$$

$$E\left[\left(\int_0^1 \kappa_T(r)\,B_T(r)^2\,dr - \int_0^1 \kappa(r)\,B(r)^2\,dr\right)^2\right] = O\left(T^{-2}\right). \tag{6}$$

Proof of (5). The (i,j)th element of Σ_{DD} is

$$\Sigma_{DD}(i,j) = \psi(1)^{-2}\,T^{-2}\sqrt{\kappa(i/T)}\sqrt{\kappa(j/T)}Cov\left(\hat{S}_i,\hat{S}_j\right).$$

Let $\psi(L) = \psi(1) + (1-L)\sum_{i=0}^{\infty}\tilde{\psi}_i L^i$ be the BN decomposition (Beveridge and Nelson (1981)) of $\psi(L)$. Elementary manipulations can be used to show that

$$|\sigma_{NN} - 1| \leq T^{-1}M_{NN}, \tag{7}$$

$$\max_{1\leq i,j\leq T-1}\left|\Sigma_{DD}(i,j) - \dot{\Sigma}_{DD}(i,j)\right| \leq T^{-2}M_{DD}\cdot\|\kappa\|, \tag{8}$$

and

$$\sigma_{DN}'\sigma_{DN} \leq T^{-2}M_{DN}\cdot\|\kappa\|, \tag{9}$$

where

$$M_{NN} = 4\psi(1)^{-2}\sum_{i=0}^{\infty}\tilde{\psi}_i^2 + |\psi(1)|^{-1}\sum_{i=0}^{\infty}\left|\tilde{\psi}_i\right|,$$

$$M_{DN} = 6\psi(1)^{-2}\sum_{i=0}^{\infty}\tilde{\psi}_i^2 + 5|\psi(1)|^{-1}\sum_{i=0}^{\infty}\left|\tilde{\psi}_i\right|,$$

and

$$M_{DD} = 9\psi(1)^{-2}\sum_{i=0}^{\infty}\tilde{\psi}_i^2 + 8|\psi(1)|^{-1}\sum_{i=0}^{\infty}\left|\tilde{\psi}_i\right|$$

are finite constants (because $\sum_{i=0}^{\infty}\tilde{\psi}_i^2 < \infty$ and $\sum_{i=0}^{\infty}\left|\tilde{\psi}_i\right| < \infty$ under A1*).

Now,

$$\left|tr\left(\Upsilon_c - \dot{\Upsilon}_c\right)\right| \leq |\sigma_{NN} - 1| + \left|tr\left(\Sigma_{DD} - \dot{\Sigma}_{DD}\right)\right| = O\left(T^{-1}\right)$$

by $(7)-(8)$, while

$$\begin{aligned} tr\left[\left(\Upsilon_c - \dot{\Upsilon}_c\right)^2\right] &\leq (\sigma_{NN}-1)^2 + 2|\sigma_{NN}-1|\,\sigma_{NN}^{-1}\sigma_{DN}'\sigma_{DN} + tr\left[\left(\Sigma_{DD} - \dot{\Sigma}_{DD}\right)^2\right] \\ &\quad +2\sigma_{NN}^{-1}\sigma_{DN}'\left(\Sigma_{DD} - \dot{\Sigma}_{DD}\right)\sigma_{DN} + 4\sigma_{NN}^{-1}\sigma_{DN}'\dot{\Sigma}_{DD}\sigma_{DN} \\ &= O\left(T^{-2}\right) + 2\sigma_{NN}^{-1}\sigma_{DN}'\left(\Sigma_{DD} - \dot{\Sigma}_{DD}\right)\sigma_{DN} + 4\sigma_{NN}^{-1}\sigma_{DN}'\dot{\Sigma}_{DD}\sigma_{DN}, \end{aligned}$$

where the equality uses $(7)-(9)$. By $(7)-(9)$ and a matrix analogue of the Cauchy-Schwarz inequality (Magnus and Neudecker (1988, Theorem 11.2)),

$$\begin{aligned} \sigma_{NN}^{-1}\sigma_{DN}'\left(\Sigma_{DD} - \dot{\Sigma}_{DD}\right)\sigma_{DN} &= \sigma_{NN}^{-1}tr\left[\left(\Sigma_{DD} - \dot{\Sigma}_{DD}\right)\sigma_{DN}\sigma_{DN}'\right] \\ &\leq \sigma_{NN}^{-1}\sigma_{DN}'\sigma_{DN}\sqrt{tr\left[\left(\Sigma_{DD} - \dot{\Sigma}_{DD}\right)^2\right]} = O\left(T^{-3}\right). \end{aligned}$$

By Magnus and Neudecker (1988, Theorem 11.4) and $(7)-(9)$,

$$\sigma_{NN}^{-1}\sigma_{DN}'\dot{\Sigma}_{DD}\sigma_{DN} \leq \lambda_{\max}\left(\dot{\Sigma}_{DD}\right)\cdot\sigma_{NN}^{-1}\sigma_{DN}'\sigma_{DN} = O\left(T^{-2}\right),$$

where $\lambda_{\max}\left(\dot{\Sigma}_{DD}\right)$ denotes the largest eigenvalue of $\dot{\Sigma}_{DD}$ and

$$\lambda_{\max}\left(\dot{\Sigma}_{DD}\right) \leq \|\kappa\| \left[2T \cdot \sin\left(\frac{\pi}{2T}\right)\right]^{-2} \leq \frac{\|\kappa\|}{4}$$

in view of Tanaka (1996, Problem 1.2.1) and the fact that $2T \cdot \sin\left(\pi/\left(2T\right)\right)$ is a decreasing function of T. Combining the results in the preceding displays and using Magnus and Neudecker (1988, Theorem 12.12),

$$E\left[\left(\eta'\Upsilon_c\eta - \eta'\dot{\Upsilon}_c\eta\right)^2\right] = \left[tr\left(\Upsilon_c - \dot{\Upsilon}_c\right)\right]^2 + 2tr\left[\left(\Upsilon_c - \dot{\Upsilon}_c\right)^2\right] = O\left(T^{-2}\right),$$

establishing (5).

Proof of (6). Upon adding and subtracting $\int_0^1 \kappa\left(r\right) B_T\left(r\right)^2 dr$ and using the fact that $B_T\left(r\right) = 0$ for $r < T^{-1}$, the difference

$$\int_0^1 \kappa_T\left(r\right) B_T\left(r\right)^2 dr - \int_0^1 \kappa\left(r\right) B\left(r\right)^2 dr$$

can be written as

$$\int_{T^{-1}}^1 \left[\kappa_T\left(r\right) - \kappa\left(r\right)\right] B_T\left(r\right)^2 dr + \int_0^1 \kappa\left(r\right)\left[B_T\left(r\right)^2 - B\left(r\right)^2\right] dr.$$

As a consequence, $E\left[\left(\int_0^1 \kappa_T\left(r\right) B_T\left(r\right)^2 dr - \int_0^1 \kappa\left(r\right) B\left(r\right)^2 dr\right)^2\right]$ is no greater than four times

$$E\left[\left(\int_{T^{-1}}^1 \left[\kappa_T\left(r\right) - \kappa\left(r\right)\right] B_T\left(r\right)^2 dr\right)^2\right] + E\left[\left(\int_0^1 \kappa\left(r\right)\left[B_T\left(r\right)^2 - B\left(r\right)^2\right] dr\right)^2\right].$$

The proof of (6) is completed by showing that each term in this display is $O\left(T^{-2}\right)$. First,

$$
\begin{aligned}
& E\left[\left(\int_{T^{-1}}^{1}\left[\kappa_T(r)-\kappa(r)\right] B_T(r)^2\, dr\right)^2\right] \\
= & \int_{T^{-1}}^{1}\int_{T^{-1}}^{1}\left[\kappa_T(r)-\kappa(r)\right]\left[\kappa_T(s)-\kappa(s)\right] E\left[B_T(r)^2 B_T(s)^2\right] dsdr \\
\leq & \left(\sup_{T^{-1}\leq r,s\leq 1} E\left[B_T(r)^2 B_T(s)^2\right]\right)\int_{T^{-1}}^{1}\int_{T^{-1}}^{1}\left|\kappa_T(r)-\kappa(r)\right|\left|\kappa_T(s)-\kappa(s)\right| dsdr \\
\leq & \left(\sup_{0\leq r,s\leq 1} E\left[B(r)^2 B(s)^2\right]\right)\cdot\left(\sup_{T^{-1}\leq r\leq 1}\left|\left(\kappa_T(r)-\kappa(r)\right)\right|\right)^2 \\
= & O\left(T^{-2}\right),
\end{aligned}
$$

where the last equality uses $\sup_{0\leq r,s\leq 1} E\left[B(r)^2 B(s)^2\right]<\infty$ and A2.

Second, consider

$$
\begin{aligned}
& E\left[\left(\int_{0}^{1}\kappa(r)\left[B_T(r)^2-B(r)^2\right] dr\right)^2\right] \\
= & 2\int_0^1\int_0^r \kappa(r)\kappa(s) E\left[\left(B_T(r)^2-B(r)^2\right)\left(B_T(s)^2-B(s)^2\right)\right] dsdr \\
\leq & 2\left\|\kappa\right\|^2\int_0^1\int_0^r \left|E\left[\left(B_T(r)^2-B(r)^2\right)\left(B_T(s)^2-B(s)^2\right)\right]\right| dsdr.
\end{aligned}
$$

Using the relation

$$
E\left[B(r)^2 B(s)^2\right] = r(1-r)s(1-s)+2\min(r,s)^2\left(1-\max(r,s)\right)^2,
$$

the integrand on the last line can be bounded as follows:

$$
\sup_{0\leq r\leq 1}\ \sup_{0\leq s\leq T^{-1}\lfloor Tr\rfloor}\left|E\left[\left(B_T(r)^2-B(r)^2\right)\left(B_T(s)^2-B(s)^2\right)\right]\right|\leq 17T^{-2}
$$

and

$$\sup_{0\leq r\leq 1}\sup_{T^{-1}\lfloor Tr\rfloor\leq s\leq r}\left|E\left[\left(B_T(r)^2-B(r)^2\right)\left(B_T(s)^2-B(s)^2\right)\right]\right|\leq 8T^{-1}.$$

As a consequence,

$$\begin{aligned}
&\int_0^1\int_0^r\left|E\left[\left(B_T(r)^2-B(r)^2\right)\left(B_T(s)^2-B(s)^2\right)\right]\right|dsdr\\
\leq\ &\sup_{0\leq r\leq 1}\int_0^r\left|E\left[\left(B_T(r)^2-B(r)^2\right)\left(B_T(s)^2-B(s)^2\right)\right]\right|ds\\
\leq\ &\sup_{0\leq r\leq 1}\int_0^{T^{-1}\lfloor Tr\rfloor}\left|E\left[\left(B_T(r)^2-B(r)^2\right)\left(B_T(s)^2-B(s)^2\right)\right]\right|ds\\
&+\sup_{0\leq r\leq 1}\int_{T^{-1}\lfloor Tr\rfloor}^r\left|E\left[\left(B_T(r)^2-B(r)^2\right)\left(B_T(s)^2-B(s)^2\right)\right]\right|ds\\
\leq\ &17T^{-2}+8T^{-2},
\end{aligned}$$

so $E\left[\left(\int_0^1\kappa(r)\left[B_T(r)^2-B(r)^2\right]dr\right)^2\right]=O\left(T^{-2}\right)$, as was to be shown. ■

Proof of Theorem 3. Under the assumptions of Theorem 3, the following convergence results hold jointly (cf. KVB):

$$T^{1/2}\left(\hat{\beta}-\beta\right)\to_d\left(Q^{-1}\Omega Q^{-1}\right)^{1/2}W_k(1) \tag{10}$$

and

$$T^{-1/2}\hat{S}_{\lfloor T\cdot\rfloor}\to_d\Omega^{1/2}B_k(\cdot),$$

where W_k is a k-dimensional Wiener process and $B_k(r)=W_k(r)-rW_k(1)$. Now,

$$\hat{\Omega}_\kappa = T^{-2} \sum_{t=1}^{T-1} \kappa\left(\frac{t}{T}\right) \hat{S}_t \hat{S}_t'$$

$$= \int_0^1 \kappa\left(\frac{\lfloor Tr \rfloor}{T}\right) \left(T^{-1/2} \hat{S}_{\lfloor Tr \rfloor}\right) \left(T^{-1/2} \hat{S}_{\lfloor Tr \rfloor}\right)' dr \tag{11}$$

$$\to_d \Omega^{1/2} \left(\int_0^1 \kappa(r) B_k(r) B_k(r)' dr\right) \Omega^{1/2\prime}$$

jointly with (10), where the last convergence result uses A2 and the continuous mapping theorem (CMT). Moreover,

$$R\left(Q^{-1}\Omega Q^{-1}\right)^{1/2} W_k(\cdot) =_d \left(RQ^{-1}\Omega Q^{-1}R'\right)^{1/2} W_q(\cdot),$$

where W_q is a q-dimensional Wiener process. As a consequence,

$$T^{1/2} R\left(\hat{\beta} - \beta\right) \to_d \left(RQ^{-1}\Omega Q^{-1}R'\right)^{1/2} \left(W_q(1) + \delta\right)$$

and

$$R\hat{Q}^{-1}\hat{\Omega}_\kappa \hat{Q}^{-1}R' \to_d \left(RQ^{-1}\Omega Q^{-1}R'\right)^{1/2} \left(\int_0^1 \kappa(r) B_q(r) B_q(r)' dr\right) \left(RQ^{-1}\Omega Q^{-1}R'\right)^{1/2}$$

jointly, where $B_q(r) = W_q(r) - rW_q(1)$ and the last convergence result uses A1(ii). Theorem 3 now follows because

$$F_\kappa = T\left(R\hat{\beta} - r\right)' \left(R\hat{Q}^{-1}\hat{\Omega}_\kappa \hat{Q}^{-1}R'\right)^{-1} \left(R\hat{\beta} - r\right)$$

$$\to_d \left(W_q(1) + \delta\right)' \left(\int_0^1 \kappa(r) B_q(r) B_q(r)' dr\right)^{-1} \left(W_q(1) + \delta\right),$$

where $W_q(1) \sim \mathcal{N}(0, I_q)$ is independent of $B_q(\cdot)$. ■

Proof of Corollary 4. It suffices to show that $\hat{\Omega}_{\kappa}^{PW} = \hat{\Omega}_{\kappa} + o_p(1)$. Under the assumptions of Corollary 4,

$$\begin{aligned} T^{-1/2}\tilde{S}_{\lfloor T\cdot \rfloor} &= T^{-1/2}\left(\hat{S}_{\lfloor T\cdot \rfloor} - \hat{A}\hat{S}_{\lfloor T\cdot \rfloor -1}\right) \\ &= \left(I_k - \hat{A}\right)T^{-1/2}\hat{S}_{\lfloor T\cdot \rfloor} + \hat{A}T^{-1/2}\left(\hat{S}_{\lfloor T\cdot \rfloor} - \hat{S}_{\lfloor T\cdot \rfloor -1}\right) \\ &= \left(I_k - A\right)T^{-1/2}\hat{S}_{\lfloor T\cdot \rfloor} + o_p(1), \end{aligned}$$

where the last equation uses A1-A2 and Billingsley (1999, Theorem 13.4). As a consequence,

$$\begin{aligned} T^{-2}\sum_{t=1}^{T-1}\kappa\left(\frac{t}{T}\right)\tilde{S}_t\tilde{S}_t' &= \left(I_k - A\right)\left(T^{-2}\sum_{t=1}^{T-1}\kappa\left(\frac{t}{T}\right)\hat{S}_t\hat{S}_t'\right)\left(I_k - A\right)' + o_p(1) \\ &= \left(I_k - A\right)\hat{\Omega}_{\kappa}\left(I_k - A\right)' + o_p(1), \end{aligned}$$

so

$$\hat{\Omega}_{\kappa}^{PW} = \left(I_k - \hat{A}\right)^{-1}\left(T^{-2}\sum_{t=1}^{T-1}\kappa\left(\frac{t}{T}\right)\tilde{S}_t\tilde{S}_t'\right)\left(I_k - \hat{A}'\right)^{-1} = \hat{\Omega}_{\kappa} + o_p(1). \qquad \blacksquare$$

Proof of Lemma 5. For any $h > 0$, it follows from (11) and CMT that

$$\begin{aligned} \hat{\Omega}_{\kappa,h} &= \min\left(\max\left[\hat{\Omega}_{\kappa}, -h\iota_k\iota_k'\right], h\iota_k\iota_k'\right) \\ &\to_d \min\left(\max\left[\Omega^{1/2}\left(\int_0^1 \kappa(r)B_k(r)B_k(r)'\,dr\right)\Omega^{1/2\prime}, -h\iota_k\iota_k'\right], h\iota_k\iota_k'\right) \end{aligned}$$

as $T \to \infty$. Moreover, it follows from CMT that

$$\min\left(\max\left[\Omega^{1/2}\left(\int_0^1 \kappa\left(r\right)B_k\left(r\right)B_k\left(r\right)'dr\right)\Omega^{1/2\prime}, -h\iota_k\iota_k'\right], h\iota_k\iota_k'\right)$$

$$\to_d \Omega^{1/2}\left(\int_0^1 \kappa\left(r\right)B_k\left(r\right)B_k\left(r\right)'dr\right)\Omega^{1/2\prime}$$

as $h\to\infty$. Repeated application of Billingsley (1999, Theorem 3.5) therefore yields

$$\begin{aligned}
&\lim_{h\to\infty}\lim_{T\to\infty}E\left[\hat{\Omega}_{\kappa,h}\right]\\
=&\lim_{h\to\infty}E\left[\min\left(\max\left[\Omega^{1/2}\left(\int_0^1 \kappa\left(r\right)B_k\left(r\right)B_k\left(r\right)'dr\right)\Omega^{1/2\prime}, -h\iota_k\iota_k'\right], h\iota_k\iota_k'\right)\right]\\
=&E\left[\Omega^{1/2}\left(\int_0^1 \kappa\left(r\right)B_k\left(r\right)B_k\left(r\right)'dr\right)\Omega^{1/2\prime}\right]
\end{aligned}$$

and

$$\lim_{h\to\infty}\lim_{T\to\infty}Var\left[vec\left(\hat{\Omega}_{\kappa,h}\right)\right]=Var\left[vec\left(\Omega^{1/2}\left[\int_0^1 \kappa\left(r\right)B_k\left(r\right)B_k\left(r\right)'dr\right]\Omega^{1/2\prime}\right)\right].$$

Now,

$$E\left[\Omega^{1/2}\left(\int_0^1 \kappa\left(r\right)B_k\left(r\right)B_k\left(r\right)'dr\right)\Omega^{1/2\prime}\right]=\Omega\int_0^1\kappa\left(r\right)m\left(r\right)dr,$$

because $E\left(B_k\left(r\right)B_k\left(r\right)'\right)=m\left(r\right)I_k$. Similarly,

$$\begin{aligned}
&Var\left[vec\left(\Omega^{1/2}\left[\int_0^1 \kappa\left(r\right)B_k\left(r\right)B_k\left(r\right)'dr\right]\Omega^{1/2\prime}\right)\right]\\
=&\left(\Omega^{1/2}\otimes\Omega^{1/2}\right)Var\left[vec\left(\left(\int_0^1 \kappa\left(r\right)B_k\left(r\right)B_k\left(r\right)'dr\right)\right)\right]\left(\Omega^{1/2}\otimes\Omega^{1/2}\right)'\\
=&\left(I_{k^2}+K_{k^2}\right)\left(\Omega\otimes\Omega\right)\int_0^1\int_0^1\kappa\left(r\right)C\left(r,s\right)\kappa\left(s\right)drds,
\end{aligned}$$

because

$$Cov\left(vec\left(B_k\left(r\right)B_k\left(r\right)'\right), vec\left(B_k\left(s\right)B_k\left(s\right)'\right)\right) = C\left(r,s\right)\left(I_{k^2} + K_{k^2}\right)$$

and

$$\left(\Omega^{1/2} \otimes \Omega^{1/2}\right)\left(I_{k^2} + K_{k^2}\right)\left(\Omega^{1/2} \otimes \Omega^{1/2}\right)' = \left(I_{k^2} + K_{k^2}\right)\left(\Omega \otimes \Omega\right),$$

where $Cov\left(X, Y\right) = E\left(XY'\right) - E\left(X\right)E\left(Y\right)'$ (for any random vectors X and Y) and the last display uses Magnus and Neudecker (1988, Theorem 3.9). $\blacksquare$

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8. Tables

TABLE 1

Percentiles of $F_{\kappa_{0.01}}$

q	90%	95%	97.5%	99%
1	3.613	5.529	7.590	10.78
2	7.508	10.42	13.77	18.60
3	12.18	16.61	21.30	27.99
4	17.72	23.29	29.39	38.49
5	24.39	31.35	38.70	48.92
6	31.67	40.04	48.41	60.53
7	39.99	50.29	60.64	76.07
8	49.24	61.24	73.41	89.26
9	58.68	72.33	85.06	104.7
10	69.97	85.44	100.6	121.0
11	82.01	98.89	115.8	138.9
12	94.72	113.2	132.8	156.7
13	107.9	128.3	149.3	176.2
14	123.0	146.1	170.1	199.9
15	137.6	162.3	186.9	221.7
16	153.8	180.9	206.9	238.7
17	171.0	200.8	229.5	267.1
18	188.7	220.4	250.8	288.5
19	207.4	240.9	273.6	314.4
20	228.1	264.2	296.8	340.6

TABLE 2
SIZE AND SIZE-ADJUSTED POWER: MONTE CARLO RESULTS
5% LEVEL TESTS, $T = 100$

Test Statistic	β	ρ = -0.5	-0.3	0	0.3	0.5	0.7	0.9	0.95
F_{HAC}	0	4.6	4.8	5.8	8.3	10.4	12.4	25.1	38.5
	0.1	15.8	15.9	15.7	15.3	15.1	14.6	14.4	13.7
	0.2	50.1	49.8	49.3	48.2	45.8	44.8	41.1	38.3
	0.3	82.7	83.4	83.3	82.6	78.6	77.0	70.7	66.5
	0.4	97.1	97.5	96.9	97.2	95.2	95.2	89.1	85.6
F_{KVB}	0	4.4	4.4	4.8	5.2	6.0	6.6	12.6	19.2
	0.1	12.6	13.0	12.9	12.9	13.8	13.4	13.3	12.8
	0.2	36.8	38.2	36.2	37.5	37.0	35.9	34.1	32.6
	0.3	65.6	65.4	65.1	64.6	63.4	64.0	59.1	55.5
	0.4	85.9	85.1	84.7	83.8	82.8	83.4	78.4	74.0
F_{κ}	0	2.6	3.2	5.0	6.8	9.8	12.3	23.9	34.9
	0.1	13.4	13.9	13.8	14.2	13.8	13.9	13.6	13.9
	0.2	39.8	42.3	40.5	42.4	39.5	40.4	37.5	38.1
	0.3	72.1	73.2	72.9	73.8	70.0	70.7	66.0	65.1
	0.4	91.9	92.4	92.3	92.1	89.4	90.2	85.4	84.1
F_{HAC}^{PW}	0	5.9	5.9	6.1	6.5	7.5	7.6	14.7	22.6
	0.1	16.3	16.4	15.2	15.0	15.5	14.5	13.6	13.7
	0.2	50.3	50.9	48.8	48.0	46.5	44.3	38.4	36.2
	0.3	83.1	83.8	82.4	82.5	79.6	77.3	66.8	60.7
	0.4	97.3	97.6	96.7	97.1	95.8	94.7	85.4	78.2
F_{KVB}^{PW}	0	5.0	4.3	4.9	4.3	4.8	5.4	8.8	13.3
	0.1	12.1	14.4	12.4	13.6	13.9	12.8	12.1	13.7
	0.2	35.2	38.9	35.7	39.2	37.9	35.9	33.8	33.6
	0.3	63.8	65.8	63.9	67.5	64.7	63.4	58.8	55.8
	0.4	83.5	85.2	84.1	85.9	84.3	83.9	77.1	72.1
F_{κ}^{PW}	0	4.0	3.5	4.6	4.4	5.3	5.4	6.9	10.5
	0.1	13.1	14.5	13.4	14.4	14.0	13.4	12.0	12.7
	0.2	37.6	41.7	38.5	41.6	40.0	38.3	33.1	30.2
	0.3	67.6	72.1	69.3	73.1	70.1	69.1	55.5	49.4
	0.4	88.4	91.1	90.3	92.0	89.6	88.4	73.2	63.5

9. Figures

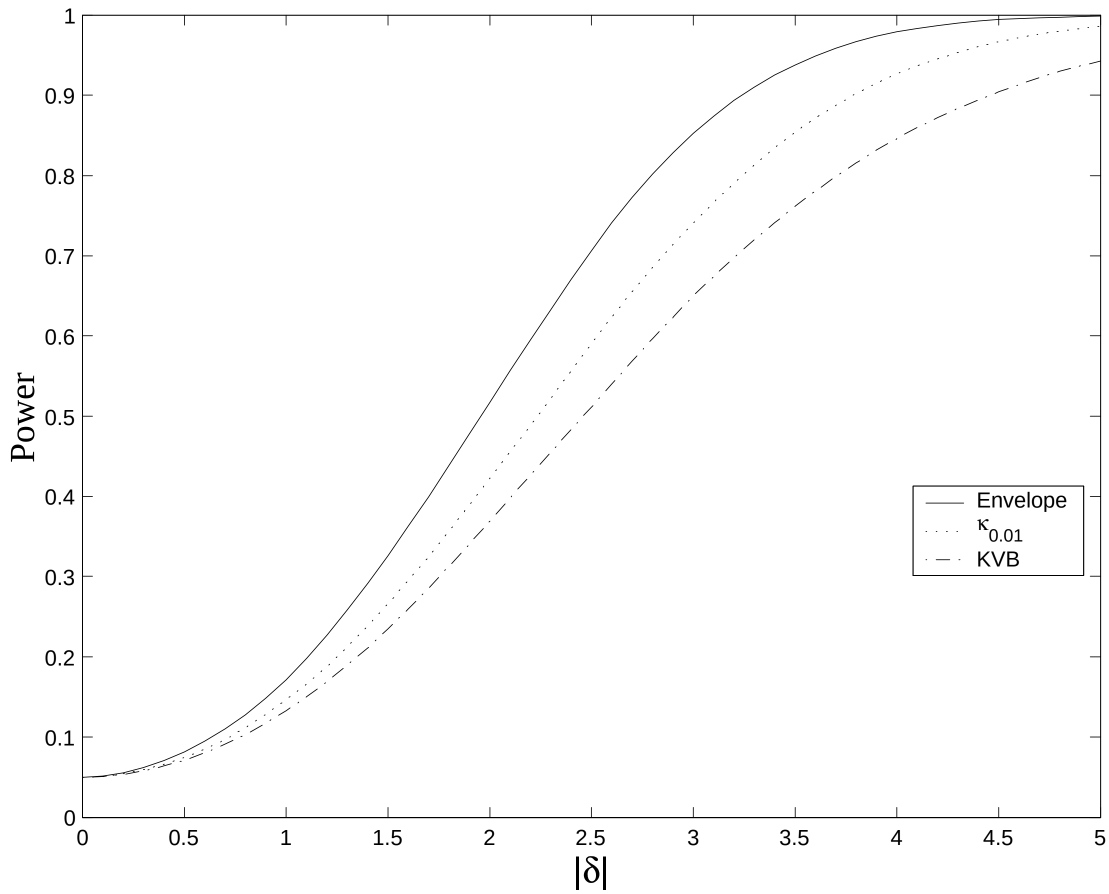

Figure 1: Local Asymptotic Power, $q = 1$.