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## core convergence

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The core of an economy is the set of all economic outcomes that cannot be ‘blocked’ by any group of individuals; it is an institution-free concept. A Walrasian equilibrium is an economic outcome based on the institution of market-clearing via prices: each individual consumes his or her demand, taking prices as given, and the demand for each good equals the supply of that good. Core convergence asserts that, for sufficiently large economies, every core allocation approximately satisfies the definition of Walrasian equilibrium; it is an important test of the price-taking assumption inherent in the definition of Walrasian equilibrium.

The core of an economy, first defined by Edgeworth (1881), is the set of all economic outcomes such that no group of individuals (‘coalition’) can make each of its members better off (‘improve on’ or ‘block’ the outcome), using only the resources available to the group. (A common mistake is to ask, in reference to a particular core allocation, ‘what coalition(s) have formed?’ An allocation is in the core precisely when no coalition can improve on it, and a core allocation does not identify an associated coalition or coalitions. It is when an allocation is *not* in the core that one can identify one or more coalitions that are associated with it, because they can improve on it and thus demonstrate that the coalition is not in the core.)

The most important reason for studying the core is the light it sheds on Walrasian equilibrium, introduced by Walras (1874). While the notion of Walrasian equilibrium is based entirely on the institution of trading via prices, and assumes that individuals take prices as given, the definition of the core is completely institution-free; this is one of its major virtues.

The core has both normative and positive significance apart from its relationship to Walrasian equilibrium. Normatively, if one accepts the distribution of the economy’s initial resources as equitable, then any allocation outside the core is unfair to at least one coalition. Regardless of whether the distribution of initial resources is equitable, it would be surprising to find the economy settling on an allocation outside the core, since that would indicate there is a coalition which *could* have made each of its members better off, using only its own resources, but for some reason has failed to coalesce and do so; this is the positive significance.

While there has been much work on the cores of production economies, the bulk of the work on the core has been carried out in exchange economies, in which trading and consuming are the only economic activities. In part, this is because there are a number of competing definitions of the core in production economies, based on how the ownership of the production technology is assigned to individuals and groups. For simplicity, we shall focus our attention on exchange economies.

Walrasian equilibrium is an economic equilibrium notion based on market clearing, mediated by prices. Consumers choose the consumption vector which maximizes utility over their budget sets; firms choose production plans which maximize profit. Critically, it is assumed that individuals and firms take prices as given, without taking into account any ability they may have to influence those prices through their actions. A price vector is a Walrasian equilibrium price if the choices made by individuals and firms, taking prices as given, are consistent in the sense that market supply equals market demand. A Walrasian allocation is the vector of individual consumptions and firm productions generated by a Walrasian equilibrium price. A Walrasian equilibrium is a pair consisting of a Walrasian equilibrium price and its associated Walrasian allocation.

An income transfer is a vector which assigns to each agent a real number, and which satisfies budget balance: the sum of the numbers is zero. An allocation is a Walrasian equilibrium with transfers if there is an income transfer and a price vector such that the demand of each agent, given the prices and the budget of the agent, taking into account the agent's endowment of goods and income transfers, just equals the individual's consumption at the allocation.

The First and Second Welfare Theorems are two of the most important results concerning Walrasian equilibrium. Recall that, in an exchange economy, an allocation is Pareto optimal if there is no reallocation of consumption which makes every agent better off. In other words, the coalition consisting of all agents (coalition of the whole) cannot improve upon the allocation. Thus, it is clear that every core allocation is Pareto optimal. The First Welfare Theorem asserts that every Walrasian allocation with transfers is Pareto optimal. A slight modification of the proof suffices to show that every Walrasian allocation lies in the core. (Note that it is *not* true that every Walrasian allocation with transfers lies in the core. The income transfers allow us to move consumption among agents. For example, consider the allocation which gives the entire social consumption to a single agent. If we choose a price vector which supports that agent's preference at the social consumption, then there is an income transfer that makes this allocation a Walrasian allocation with transfers. But this allocation will rarely lie in the core, since the coalition consisting of all the other agents will generally be able to improve on it.) This is an important strengthening of the First Welfare Theorem, which has both positive and normative significance. On the positive side, it is a strong stability property of Walrasian equilibrium, since it asserts that

no group of individuals would choose to upset the equilibrium by recontracting among themselves, making it more plausible that we will see Walrasian equilibrium arise in real economies. On the normative side, if we accept the distribution of initial endowments as equitable, it tells us that Walrasian allocations are fair to all groups in the economy.

The Second Welfare Theorem asserts that, in an exchange economy with standard assumptions on preferences (convexity is the crucial assumption), every Pareto optimal allocation is a Walrasian equilibrium with transfers. Note that while the definition of Pareto optimality makes no mention of prices, the Second Welfare Theorem asserts that every Pareto optimal allocation is closely associated to a price vector. The price vector appears magically; mathematically, this is a consequence of the separating hyperplane theorem, for which convexity is a critical assumption. As noted above, the most important use of the core is as a test of the price-taking assumption inherent in the definition of Walrasian equilibrium; a number of other tests have been proposed, but *core convergence* is the most commonly used. Core convergence is closely analogous to the conclusion of the Second Welfare Theorem. The definition of the core makes no mention of prices. However, if an exchange economy is sufficiently large, it is a remarkable fact that every core allocation is closely associated with a price vector that ‘approximately decentralizes’ it; in other words, every core allocation approximately satisfies the definition of Walrasian equilibrium, *without transfers*. This is an important strengthening of the Second Welfare Theorem. The notion of approximate decentralization depends to a considerable extent on the assumptions one is willing to make on the preferences and endowments of the individuals in the economy. (One version states that core allocations can be realized as *exact* Walrasian equilibrium with *small* income transfers.)

Core convergence has a number of implications, both normative and positive. The extent to which each of these implications is justified in a particular setting depends a great deal on the form of convergence, and thus on the assumptions one is willing to make on the economy. For an extensive survey focusing on the relationship between assumptions and the form of convergence, see Anderson (1992).

On the normative side, core convergence is a strong ‘unbiasedness’ property of Walrasian equilibrium, since it asserts that restricting attention to Walrasian allocations does not narrow the set of outcomes much beyond the narrowing that occurs in the core. Thus, Walrasian equilibrium has no hidden implications for the welfare of different groups, beyond whatever equity concerns one might have over the initial endowments. If one accepts the distribution of initial endowments as equitable, then any allocation that is far from Walrasian will not be in the core, and hence will treat some group of agents unfairly. On the positive side, if one accepts the core as a positive description of the allocations one is likely to see in practice in any economy, then core convergence tells us that the allocations we see will be nearly Walrasian.

However, the greatest significance of core convergence is as a test of the reasonableness of the price-taking assumption that is hidden in plain sight in the definition of Walrasian equilibrium. In real markets, we see prices used to equate supply and demand, but this does not guarantee Walrasian outcomes. Agents possessing market power may choose to demand quantities different from their price-taking demands at the prevailing price, thereby altering that price and leading to a non-Walrasian outcome. If the outcome is not at least approximately Walrasian, then the welfare theorems and the results on existence and generic determinacy of Walrasian allocations would have limited implications for real economies.

Core convergence and non-convergence allows us to identify situations in which price-taking is more or less reasonable. Core convergence implies that all trade takes place at almost a single price. An agent who tries to bargain cannot influence the prices much, so there is little incentive to be anything other than a price-taker. On the other hand, core non-convergence makes price-taking an implausible assumption.

Edgeworth (1881) doubted the positive significance of Walrasian equilibrium, and argued that the core, not the set of Walrasian equilibria, was the best positive description of the outcomes of a market mechanism. Moreover, while Edgeworth's name is closely associated with core convergence, and he did prove a core convergence theorem, he argued that in real economies, the presence of firms and syndicates which possess market power ensures that the core does *not* converge.

Edgeworth's argument about the effects of market power applies most strongly to the production side of the economy, where we do in fact see large firms, syndicates and labour unions. However, on the consumption side, the wealthiest individual in the world consumes a small part of the world's annual consumption. In exchange economies in which each consumer is small, core convergence holds. So core convergence provides a justification for the price-taking assumption on the consumption side, provided one views the world as an exchange economy in which the production decisions have been previously made by some exogenous process, outside the scope of the model, endowments include the income obtained by selling one's labour in the exogenous production process, and the only economic activity is trade and consumption of what has been produced.

The proof of the most basic core convergence theorem, which assumes very little about preferences and endowments, and establishes approximate decentralization in a relatively weak sense, is closely analogous to the proof of the Second Welfare Theorem. The approximately decentralizing price vector appears magically, as a consequence of the separating hyperplane theorem and the Shapley–Folkman theorem, which asserts that the sum of a large number of sets is approximately convex. Convexity of preferences plays no role. Indeed, the definition of the core, because it allows for individuals to be included or excluded from potential coalitions,

introduces a non-convexity which is not present in the Second Welfare Theorem, and the Shapley–Folkman theorem controls that non-convexity, whether or not preferences themselves are convex.

The definitions and results just described verbally are presented more formally below.

Many people have made important contributions to the study of core convergence. A survey of these contributions is given in Anderson (1992), and a list of some of the more important contributions is included in the bibliography.

Now, we turn to a more formal presentation.

**Definition 1** In an exchange economy with agents  $i = 1, \dots, I$  having strict preferences  $\succsim_i$  and endowments  $\omega_i \in \mathbf{R}_+^L$ , a *coalition* is a set  $S \subseteq \{1, \dots, I\}$ . An *exact allocation* is  $x \in (\mathbf{R}_+^L)^I$  such that  $\sum_{i=1}^I x_i = \sum_{i=1}^I \omega_i$ . An exact allocation is *weakly Pareto optimal* if there is no other exact allocation  $x'$  satisfying  $x'_i \succsim_i x_i (i = 1, \dots, I)$ . A coalition  $S$  *blocks* or *improves on* an exact allocation  $x$  by  $x'$  if  $\sum_{i \in S} x'_i = \sum_{i \in S} \omega_i$  and  $\forall_{i \in S} x'_i \succsim_i x_i$ . The *core* is the set of all exact allocations which cannot be improved on by any nonempty coalition. The *price simplex* is  $\Delta = \{p \in \mathbf{R}_+^k : \sum_{\ell=1}^L p_\ell = 1\}$ .

**Theorem 2** *In an exchange economy, every core allocation is weakly Pareto optimal.*

**Proof:** If  $x$  is not weakly Pareto optimal, then there exists  $x'$  such that  $\sum_{i=1}^I x'_i = \sum_{i=1}^I x_i$ ,  $x'_i \succsim_i x_i$ . Then  $S = \{1, \dots, I\}$  improves on  $x$  by  $x'$ , so  $x$  is not in the core.

**Theorem 3 (Strong First Welfare Theorem)** *In an exchange economy, every Walrasian Equilibrium lies in the core.*

**Proof:** Suppose  $(p^*, x^*)$  is a Walrasian Equilibrium. If  $x^*$  is not in the core, there exists  $S \subseteq I$ ,  $S \neq \emptyset$  and  $x'_i (i \in S)$  such that  $\sum_{i \in S} x'_i = \sum_{i \in S} \omega_i$  and  $x'_i \succsim_i x_i^*$  for each  $i \in S$ . Since  $x_i^*$  lies in  $i$ 's demand set at the price  $p^*$ ,  $p^* \cdot x_i^* > p^* \cdot \omega_i$ , so  $p^* \cdot \sum_{i \in S} x'_i = \sum_{i \in S} p^* \cdot x'_i > \sum_{i \in S} p^* \cdot \omega_i = p^* \cdot \sum_{i \in S} \omega_i$  but  $\sum_{i \in S} x'_i = \sum_{i \in S} \omega_i$ , a contradiction. Therefore,  $x^*$  is in the core. ♦

**Theorem 4 (Core Convergence, E. Dierker, 1975, and Anderson, 1978)** *Suppose we are given an exchange economy with  $L$  commodities,  $I$  agents, and preferences  $\succsim_1, \dots, \succsim_I$  satisfying weak monotonicity (if  $x \gg y$ , then  $x \succsim_i y$ ) and the following free disposal condition:  $x \gg y, y \succsim_i z \Rightarrow x \succsim_i z$ . If  $x$  is in the core, then there exists  $p$*

$\in \Delta$  such that

$$\frac{1}{I} \sum_{i=1}^I |p \cdot (x_i - \omega_i)| \leq \frac{2L}{I} \max\{\|\omega_1\|_\infty, \dots, \|\omega_I\|_\infty\} \quad (1)$$

$$\frac{1}{I} \sum_{i=1}^I |\inf\{p \cdot (y - x_i) : y \succ_i x_i\}| \leq \frac{4L}{I} \max\{\|\omega_1\|_\infty, \dots, \|\omega_I\|_\infty\} \quad (2)$$

where  $\|x\|_\infty = \max\{|x_1|, \dots, |x_L|\}$ .

If there are many more agents than goods, and the endowments are not too large, the bounds on the right hand sides of eqs (1) and (2) will be small. In that case, eq. (1) says that trade occurs almost at the price  $p$ , and that each  $x_i$  is almost in the budget set, while eq. (2) says that the price  $p$  almost supports  $\succ_i$  at  $x_i$ , in the sense that everything preferred to  $x_i$  costs almost as much as  $x_i$ . Taken together, eqs (1) and (2) say that the pair  $(p, x)$  satisfies a slightly perturbed version of the *definition* of Walrasian equilibrium. Indeed, if we knew the left sides of eqs (1) and (2) were *zero*, then  $p \cdot (x_i - \omega_i) = 0$ , so  $x_i$  lies in  $i$ 's budget set, and  $y \succ_i x_i \Rightarrow p \cdot y \geq p \cdot \omega_i$ , so  $x$  would be a Walrasian quasi-equilibrium! (A pair  $(p^*, x^*)$  is said to be a Walrasian quasi-equilibrium if it satisfies the definition of a Walrasian equilibrium except that instead of requiring that  $x_i^*$  lie in  $i$ 's demand set, we only require that  $x_i^*$  lie in  $i$ 's quasi-demand set, that is  $p^* \cdot x_i^* \leq p^* \cdot \omega_i$  and every  $y \succ_i x_i^*$  satisfies  $p^* \cdot y \geq p^* \cdot \omega_i$ .)

*Outline of Proof:* Follow the proof of the Second Welfare Theorem.

- Suppose  $x$  is in the core. Define  $B_i = \{y - \omega_i : y \succ_i x_i\} \cup \{0\} = (\{y : y \succ_i x_i\} \cup \{\omega_i\}) - \omega_i$  and  $B = \sum_{i=1}^I B_i$ .

The first term in the definition of  $B_i$  corresponds to members of a potential improving coalition; for accounting purposes, we assign members outside the coalition their endowments. Note that  $B_i$  is *not* convex, even if  $\succ_i$  is a convex preference.

*Claim:* If  $x$  is in the core, then  $B \cap \mathbf{R}_{--}^L = \emptyset$ . Suppose  $z \in B \cap \mathbf{R}_{--}^L$ . Then there exists  $Z_i \in B_i$  such that

$z = \sum_{i=1}^I z_i$ . Let  $S = \{i : z_i \neq 0\}$ ; since  $z \ll 0$ ,  $S \neq \emptyset$ . For  $i \in S$ , let  $x'_i = \omega_i + z_i - \frac{z}{|S|}$ . Then

$x'_i \gg \omega_i + z_i \succ_i x_i$  by the definition of  $B_i$ ,  $x'_i \succ_i x_i$  by free disposal, and  $\sum_{i \in S} x'_i = \sum_{i \in S} \omega_i$ , so  $S$  can

improve on  $x$  by  $x'$ , so  $x$  is not in the core.

- Let  $v = -L(\max_{i=1, \dots, I} \|\omega_i\|_\infty, \dots, \max_{i=1, \dots, I} \|\omega_i\|_\infty)$ . *Claim:*  $(\text{con } B) \cap (v + \mathbf{R}_{--}^L) = \emptyset$ .

If  $Z \in \text{con}B$ , by the Shapley–Folkman theorem, and relabelling the agents, we may write

$$z = \sum_{i=1}^I z_i, \quad z_i \in \text{con } B_i \quad (i = 1, \dots, I), \quad z_i \in B_i \quad (i \neq \{1, \dots, L\}).$$

Choose

$$\hat{z}_i = \begin{cases} 0 & \text{if } i = 1, \dots, L \\ z_i & \text{if } i = L+1, \dots, I \end{cases}$$

Then  $\sum_{i=1}^I \hat{z}_i \in B$  so  $\sum_{i=1}^I \hat{z}_i \ll 0$ . If  $z \ll v$ , then

$$\sum_{i=1}^I \hat{z}_i = \sum_{i=1}^L 0 + \sum_{i=L+1}^I z_i \leq \sum_{i=1}^L (\omega_i + z_i) + \sum_{i=L+1}^I z_i = \sum_{i=1}^L \omega_i + \sum_{i=1}^I z_i = \sum_{i=1}^L \omega_i + z \ll \sum_{i=1}^L \omega_i + v \leq 0,$$

so  $B \cap \mathbf{R}_{--}^L \neq \emptyset$ , a contradiction which proves the claim.

- By the separating hyperplane theorem, there exists  $p \neq 0$  such that  $\sup p \cdot (v + \mathbf{R}_{--}^L) \leq \inf p \cdot (\text{con } B)$ . If  $p_\ell < 0$  for some  $\ell$ , then  $\sup p \cdot (v + \mathbf{R}_{--}^L) = +\infty$ , while  $\inf p \cdot (\text{con } B) \leq 0$ , a contradiction, so  $p > 0$  and we can normalize  $p \in \Delta$ . Then  $\inf p \cdot B \geq \inf p \cdot (\text{con } B) \geq p \cdot v = -L \max \{\|\omega_1\|_\infty, \dots, \|\omega_l\|_\infty\}$ .
- Adapt the remainder of the proof of the Second Welfare Theorem; this requires a few tricks.

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*See also* Arrow–Debreu model of general equilibrium; cores; Edgeworth, Francis Ysidro; existence of general equilibrium; general equilibrium; general equilibrium (recent developments)

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