Math 104–Spring 2005–Anderson
Lecture Notes on Contraction Mapping Theorem

Definition 0.1 Let (S, d) be a metric space. A function $f : S \to S$ is a *contraction* if

$$\exists \alpha \in [0,1)\ \forall x, y \in S\ d(f(x), f(y)) \leq \alpha d(x, y)$$

s is a *fixed point* of f if $f(s) = s$.

Theorem 0.2 (Contraction Mapping Theorem) *If (S, d) is a complete metric space, and $f : S \to S$ is a contraction, then f has a unique fixed point.*

Proof: We first show that a fixed point exists. Since $S \neq \emptyset$, we may choose an arbitrary $s_0 \in S$. Consider the sequence (s_n) defined by

$$\begin{aligned} s_1 &= f(s_0) \\ s_2 &= f(s_1) \\ &\vdots \\ s_{n+1} &= f(s_n) \end{aligned}$$

If $s_1 = s_0$, then $f(s_0) = s_1 = s_0$, so s_0 is a fixed point.

If $s_1 \neq s_0$, we claim that

$$d(s_{n+1}, s_n) \leq \alpha^n d(s_1, s_0)$$

The proof of the claim is by induction. Note that

$$\begin{aligned} d(s_2, s_1) &= d(f(s_1), f(s_0)) \\ &\leq \alpha d(s_1, s_0) \end{aligned}$$

Now suppose that $d(s_{n+1}, s_n) \leq \alpha^n d(s_1, s_0)$. Then

$$\begin{aligned} d(s_{n+2}, s_{n+1}) &= d(f(s_{n+1}), f(s_n)) \\ &\leq \alpha d(s_{n+1}, s_n) \\ &\leq \alpha \cdot \alpha^n d(s_1, s_0) \\ &= \alpha^{n+1} d(s_1, s_0) \end{aligned}$$

so the claim follows by induction.

Next, we show that (s_n) is a Cauchy sequence. Let $\varepsilon > 0$. Since $0 < \alpha < 1$, $\lim_{n\to\infty} \alpha^n = 0$ by Example 9.7(b). Choose N such that

$$\alpha^N < \frac{\varepsilon(1-\alpha)}{d(s_1, s_0)}$$

If $m \geq n > N$,

$$\begin{aligned} d(s_m, s_n) &\leq d(s_m, s_{m-1}) + \cdots + d(s_{n+1}, s_n) \\ &< \alpha^{m-1} d(s_1, s_0) + \alpha^{m-2} d(s_1, s_0) + \cdots + \alpha^n d(s_1, s_0) \\ &= \left(\frac{1-\alpha^m}{1-\alpha} - \frac{1-\alpha^n}{1-\alpha}\right) d(s_1, s_0) \text{ (by Exercise 9.18)} \\ &= \frac{\alpha^n - \alpha^m}{1-\alpha} d(s_1, s_0) \\ &< \frac{\alpha^n}{1-\alpha} d(s_1, s_0) \\ &< \frac{\alpha^N}{1-\alpha} d(s_1, s_0) \\ &< \varepsilon \end{aligned}$$

so (s_n) is Cauchy.

Since (S, d) is complete, (s_n) has a limit $s \in S$. We will show that $f(s) = s$. Fix $\varepsilon > 0$. There exists N_1 such that

$$n > N_1 \Rightarrow d(s_n, s) < \frac{\varepsilon}{2}$$

Since (s_n) is Cauchy, there exists N_2 such that

$$n > N_2 \Rightarrow d(s_{n+1}, s_n) < \frac{\varepsilon}{2}$$

Choose any $n > \max\{N_1, N_2\}$. Then

$$\begin{aligned} d(s, f(s)) &\leq d(s, s_{n+1}) + d(s_{n+1}, f(s)) \\ &= d(s, s_{n+1}) + d(f(s_n), f(s)) \\ &< \frac{\varepsilon}{2} + \alpha d(s_n, s) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\ &= \varepsilon \end{aligned}$$

Since ε is arbitrary, $d(s, f(s)) = 0$, so $f(s) = s$. Thus, f has a fixed point.

To show the fixed point is unique, suppose $f(s) = s$ and $f(t) = t$. Then

$$\begin{aligned} d(s, t) &= d(f(s), f(t)) \\ &\leq \alpha d(s, t) \end{aligned}$$

so

$$(1 - \alpha) d(s, t) \leq 0$$

so $d(s, t) \leq 0$. Since d is a metric, $d(s, t) = 0$, and thus $s = t$, so the fixed point is unique. ■