Econ 204
Set Formation and the Axiom of Choice

In this supplement, we discuss the rules underlying set formation and the Axiom of Choice.

We generally begin with a set of elements, such as the natural numbers $\mathbf{N}$, the rational numbers $\mathbf{Q}$, the real numbers $\mathbf{R}$, or an abstract set like the set X of all points of an unspecified metric space.

Given any set X, we can form 2^X, often called the *power set* of X; 2^X is the set of all subsets of X. Thus, we can form the set $\mathbf{N}$ of all natural numbers, $2^{\mathbf{N}}$, the set of all subsets of $\mathbf{N}$; $\emptyset$, $\{1,2\}$, $\{2,4,6,\ldots\}$ are elements of $2^{\mathbf{N}}$.

We can also form $2^{2^{\mathbf{N}}} = 2^{\left(2^{\mathbf{N}}\right)}$, the set of all subsets of the set of all subsets of the natural numbers. An element of $2^{2^{\mathbf{N}}}$ is a set of subsets of the natural numbers; for example, $\{\emptyset\}$, $\{\emptyset,\mathbf{N}\}$, $\{\{1\},\{2\},\{2,4,6,\ldots\}\}$ and $\{\{2\},\{4\},\{6\},\ldots\}$ are elements of $2^{2^{\mathbf{N}}}$.

Let X be any set, and $P(x)$ a mathematical statement about a variable x. Then

$$\{x \in X : P(x)\}$$

is a set; it is the collection of all elements x of X such that the statement $P(x)$ is true. For example, if f is a function from $[a,b]$ to $\mathbf{R}$, then $\{t \in [a,b] : f(t) < 7\}$ is a valid set; it consists of all those elements t in the interval $[a,b]$ such that $f(t) < 7$. The statement P can be complex. In particular, it can include quantifiers. For example,

$$\{x \in [0,1] : \forall_{y \in [0,1]}\, x \geq y\}$$

is a valid set; it equals $\{1\}$.

$$\{x \in (0,1) : \forall_{y \in (0,1)}\, x > y\}$$

is also a valid set; it equals the empty set. The set of all upper bounds for $X \subseteq 2^{\mathbf{R}}$ is

$$U = \{u \in \mathbf{R} : \forall_{x \in X} u \geq x\}$$

In order to avoid Russell's Paradox[1], one needs to exercise a little care in forming sets. In practice, the things that a working economist needs to do are always legal. You can always apply the power set construction an arbitrary finite number of times, and use quantifiers of the form $\forall_{x \in X}$ as long as X is a set formed by taking at most a finite number of applications of the power set operation. Thus,

$$2^{\left(2^{\left(2^{\mathbf{R}}\right)}\right)}$$

is fine. Working economists have no interest in sets[2] like

$$Y = \{1, \{1\}, \{\{1\}\}, \{\{\{1\}\}\}, \ldots\}$$

which involve unbounded applications of the power set construction.

A function $f : X \to Y$ is defined in terms of its graph

$$G_f = \{(x, y) : y = f(x)\} \subseteq X \times Y = \{(x, y) : x \in X, y \in Y\}$$

so $G_f \in 2^{X \times Y}$. The fact that f is a function says that

$$((x, y) \in G_f \wedge (x, z) \in G_f) \Rightarrow (y = z)$$

The collection of all functions mapping X to Y is thus a subset of $2^{X \times Y}$, and is thus an element of $2^{2^{X \times Y}}$. Therefore, we can write quantifiers over functions. For example,

$$\forall_{f:\mathbf{N} \to \mathbf{R}} \exists_{x \in \mathbf{R}} \ \nexists n \in \mathbf{N}\, f(n) = x$$

states that there is no function mapping $\mathbf{N}$ onto $\mathbf{R}$.

Suppose we are given a set Λ and a function $G : \Lambda \to 2^X$ for some set X. Then the Axiom of Choice asserts

$$(\forall_{\lambda \in \Lambda}\, G(\lambda) \neq \emptyset) \Rightarrow (\exists_{f:\Lambda \to X} \forall_{\lambda \in \Lambda}\, f(x) \in G(\lambda))$$

[1]In the early days of set theory, mathematicians were somewhat cavalier about what constituted a set. Bertrand Russell point out that if the collection of all sets is a set Ω, then one can form $E = \{X \in \Omega : X \notin X\}$, the set of all sets which are not elements of themselves. Is $E \in E$? If so, then $E \notin E$, contradiction; if not, then $E \in E$, again a contradiction. Thus, one needs to define the notion of "set" in such a way that the collection of all sets is not a "set."

[2]Y is a valid set, but one needs to exercise caution with respect to quantifiers of the form $\forall_{y \in Y}$.

In other words, if I can chose an element of $G(\lambda)$ one λ at a time, I can choose a *function* $f : \Lambda \to X$ such that $f(\lambda) \in G(\lambda)$ for all $\lambda \in \Lambda$. For example, suppose that for all $n \in \mathbf{N}$, $B_{1/n}(y) \cap X \neq \emptyset$. Then the Axiom of Choice tells us that there is a sequence (recall a sequence is a function whose domain is $\mathbf{N}$) $\{x_n\}$ of elements of x such that $x_n \in B_{1/n}(y)$ (and hence $x_n \to y$).