Section 6
Econ 204, GSI: Hui Zheng

Key words

Extreme Value Theorem, Intermediate Value Theorem, Monotonically Increasing, Completeness, Contraction Mapping Theorem, Cauchy Sequence

Section 6.1 Properties of Continuous Functions

- Lecture 5 Theorem 1 (Extreme Value Theorem): Let f be a continuous real-valued function on $[a, b]$. Then f assumes its minimum and maximum on $[a, b]$. In particular, f is bounded above and below.
- Lecture 5 Theorem 2 (Intermediate Value Theorem): Suppose $f : [a, b] \to R$ is continuous, and $f(a) < d < f(b)$. Then there exists $c \in (a, b)$ such that $f(c) = d$.
  Example 6.1.1
  If f is continuous real-valued function on $[a, b]$, then $f([a, b])$ is a closed interval.
  Solution: The Extreme Value Theorem shows that the range of f is bounded, and the extrema are attained. Thus there are points c and d in $[a, b]$ such that

$$\begin{aligned} f(c) &= m := \inf_{x\in[a,b]} f(x) \\ f(d) &= M := \sup_{x\in[a,b]} f(x) \end{aligned}$$

  Suppose that $c \leq d$ (the case $c > d$ may be handled similarly by considering the function $-f$). Pick any arbitrary point y in (m, M). Since $f(x)$ is continuous on the interval $[c, d]$. Thus by the Intermediate Value Theorem, there is a point $x \in (c, d)$ such that $f(x) = y$. This is true for every point $y \in (m, M)$, as well as two end points. Therefore, $f([a, b]) = [m, M]$.

Section 6.2 Cauchy Sequence

- Lecture 5 Definition 6: A sequence $\{x\}$ in a metric space (X, d) is **Cauchy** if $\forall \varepsilon > 0\, \exists N(\varepsilon)\, n, m > N(\varepsilon) \Rightarrow d(x_n, x_m) < \varepsilon$
- Lecture 5 Theorem 7: Every convergent sequence in a metric space is Cauchy.

Example 6.2.1

Show that the sequence $\{x_n\} = \frac{(-1)^n}{n}$ is Cauchy with Euclidean metric.

Solution 1: Pick an $\varepsilon > 0$, choose $N > \frac{2}{\varepsilon}$, for every $m, n > N$, $|x_m - x_n| < |\frac{(-1)^m}{m} - \frac{(-1)^n}{n}| \leq |\frac{1}{m} + \frac{1}{n}| < \frac{2}{N} < \varepsilon$. So it is Cauchy.

Solution 2: We know that $\{x_n\} \to 0$. All convergent sequences are Cauchy (Theorem 7). So $\{x_n\}$ is Cauchy.

Example 6.2.2

Show that if x_n and y_n are Cauchy sequences from a metric space X, then $d(x_n, y_n)$ converges.

Solution:

Because X is not necessarily complete, we cannot rely on the convergence of x_n and y_n. The fact that the sequences are Cauchy means that for all $\varepsilon > 0$, there exists an $N_x(\varepsilon)$ such that $m, n \geq N_x(\varepsilon) \Rightarrow d(x_m, x_n) < \varepsilon$ and there exists an $N_y(\varepsilon)$ such that $m, n \geq N_y(\varepsilon) \Rightarrow d(y_m, y_n) < 0$. We will use this to show that the sequence $d(x_n, y_n)$ is Cauchy.Then because $d(x_n, y_n)$ is in $\mathbf{R}$ and $\mathbf{R}$ is complete, it must converge.

First let us make note of two facts which come from repeated application of the triangle inequality:

$$\begin{aligned} d(x_n, y_n) &\leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) \\ d(x_m, y_m) &\leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m) \end{aligned}$$

Rearranging these (by isolating the expression $d(x_m, y_m) - d(x_n, y_n)$) yields

$$\begin{aligned} -(d(x_m, x_n) + d(y_m, y_n)) &\leq d(x_m, y_m) - d(x_n, y_n) \leq d(x_m, x_n) + d(y_m, y_n) \\ &\Downarrow \\ |d(x_m, y_m) - d(x_n, y_n)| &\leq d(x_m, x_n) + d(y_m, y_n) \end{aligned}$$

Now given $\varepsilon > 0$, choose $N(\varepsilon) > \max\{N_x(\frac{\varepsilon}{2}), N_y(\frac{\varepsilon}{2})\}$. Then $\forall n \geq N(\varepsilon) \Rightarrow |d(x_m, y_m) - d(x_n, y_n)| \leq d(x_m, x_n) + d(y_m, y_n) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$. So $d(x_n, y_n)$ is Cauchy and consequently converges.

Example 6.2.3

Prove that a sequence of real numbers $\{a_n\}$ converges iff it is Cauchy.

Solution: Assume the sequence is Cauchy. Given $\varepsilon > 0$, there exists $N(\varepsilon)$ such that $|a_n - a_m| < \varepsilon/2$, $\forall n, m > N(\varepsilon)$. Our first observation is that a Cauchy sequence is a bounded sequence. In fact, letting $\epsilon = 1$, put $M = max\{|a_1|, |a_2|, ..., |a_{N(1)}|, |a_{N(1)+1}| + 1/2\}$. Then, M is an upper bound on $|a_n|$, $\forall n$. Since the sequence is bounded we know, by the Bolzano-Weierstrass Theorem, that there is a subsequence a_{n_k} that converges (to some limit L). Choose an index N' such that $\forall n_k > N'$, $|a_{n_k} - L| < \varepsilon/2$. Then $\forall n, n_k > N = max\{N(\varepsilon), N'\}$

$$|a_n - L| = |a_n - a_{n_k} + a_{n_k} - L| \leq |a_n - a_{n_k}| + |a_{n_k} - L| < \varepsilon$$

Notice that the first term on the RHS is $< \epsilon/2$ by the fact that the sequence is Cauchy. The second term is $< \varepsilon/2$ because the a_{n_k}'s form a convergent subsequence. It follows that $|a_n - L| < \varepsilon \ \forall n > N$.

Now assume the sequence converges (to some number L). Given $\varepsilon > 0$ there exists $N(\varepsilon)$ such that whenever $n > N(\varepsilon)$, $|a_n - L| < \varepsilon/2$. Let $m, n > N(\varepsilon)$. We then obtain:

$$|a_n - a_m| = |a_n - L + L - a_m| \leq |a_n - L| + |L - a_m| < \varepsilon$$

By hypothesis, both terms on the RHS are $< \varepsilon/2$. Therefore, the a_n's are Cauchy.

Example 6.2.4:

Are the following statements true or false? Write down your proof or disproof.

(1) Cauchy sequence $\Rightarrow$ bounded squence

(2) bounded squence $\Rightarrow$ Cauchy sequence

(3) Convergent squence $\Rightarrow$ Cauchy sequence

(4) Cauchy sequence $\Rightarrow$ Convergent squence

Section 6.3 Complete Metric Spaces

- Lecture 5 Definition 8: A metric space (X, d) is **complete** if every Cauchy sequence $\{x_n\} \subseteq X$ converges to a limit $x \in X$.
- Lecture 5 Definition 8: A **Banach space** is a normed space which is complete in the metric generated by its norm.
- Lecture 5 Theorem 9: $\mathbf{R}$ is complete with the usual metric (so E^1 is a Banach space).
- Lecture 5 Theorem 10: E^n is complete for every $n \in \mathbf{N}$
- Lecture 5 Theorem 11: Suppose (X, d) is a complete metric space, $Y \subseteq X$. Then $(Y, d) = (Y, d|_y)$ is complete if and only if Y is a closed subset of X.

  Note

  Why we need the definition of completeness? It's because convergent point (limit point) may not exist in X. A careful examination of definition 5 and definition 8 tells us that whether some sequence is Cauchy depends on how the metric is defined rather than the underlying metric space. But whether a Cauchy sequence converges in a metric space depends on the definition of the metric space.

Example 6.3.1

Show that $(0, \infty)$ is not complete.

Solution: The easiest way to show that a metric space is not complete is to find a Cauchy sequence that converges to a limit out of the space. Pick a sequence $\{x_n\} = \frac{1}{n}$ in the metric space $(0, \infty)$. $\{x_n\}$ is a Cauchy sequence: Given an $\varepsilon > 0$, choose $N > \frac{1}{\varepsilon}$, for every $m, n > N$, $|x_m - x_n| < \max\{\frac{1}{m}, \frac{1}{n}\} < \frac{1}{N} < \varepsilon$. And we know it converges to 0. But 0 $\notin (0, \infty)$, so this metric space is not complete.

Example 6.3.2

Let X denote the set of all bounded finite and infinite sequences of real numbers $\{a_n\}_{n=1}^{\infty}$ (hereafter denoted simply as a_n). Define the "distance" between two sequences a_n and b_n to be: $d(a_n, b_n) = \sum_{n=1}^{\infty} 2^{-n}|a_n - b_n|$.

(a). Show that: (X, d) is a metric space.

(b). Show that (X, d) is not complete.

Solution:

(a). Elements of X are sequences of numbers. Put $\{a_n\}, \{b_n\}, \{c_n\} \in X$. We check only the triangle inequality as the other two properties of a metric are obviously satisfied: $d(\{a_n\}, \{b_n\}) + d(\{b_n\}, \{c_n\}) = \sum_{n=1}^{\infty} 2^{-n}|a_n - b_n| + \sum_{n=1}^{\infty} 2^{-n}|b_n - c_n| = \sum_{n=1}^{\infty} 2^{-n}(|a_n - b_n| + |b_n - c_n|) \geq \sum_{n=1}^{\infty} 2^{-n}|a_n - c_n| = d(\{a_n\}, \{c_n\})$, where the last inequality follows from the usual triangle inequality for the real numbers.

Remark: The assumptions on the space X ensure that the equation defining d in fact yields a *function* from X to $\Re$: For any $\{a_n\}, \{b_n\} \in X$, put A_1, A_2 such that $|a_n| \leq A_1, |b_n| \leq A_2$ $\forall n$. We then obtain: $d(\{a_n\}, \{b_n\}) = \sum_{n=1}^{\infty} 2^{-n}|a_n - b_n| \leq \sum_{n=1}^{\infty} 2^{-n}(|a_n| + |b_n|) \leq \sum_{n=1}^{\infty} 2^{-n}(A_1 + A_2) < \infty$.

(b). To show that X is not complete we will produce a sequence of elements $a^m = \{a_r^m\}_{r=1}^{\infty} \in X$ such that $d(a^m, a^n) \to 0$ as $m, n \to \infty$ and $a = \{a_r\}, a_r = \lim_{m\to\infty} a_r^m \notin X$. In other words, we will produce a Cauchy sequence of elements of X whose componentwise limits form an unbounded sequence. Define a^m by setting: $a_r^m = \begin{cases} r & if\, r < m \\ m & if\, r \geq m \end{cases}$

Notice that $a_r = \lim_{m\to\infty} a_r^m = r$, hence the sequence $a = \{a_r\}$ formed by the componentwise limits is unbounded and, thus, not an element of X. On the other hand, note that the sequence of elements $a^m = \{a_r^m\}$ is a Cauchy sequence in X: Given any $\epsilon > 0$ we will show that there exists an $N(\epsilon)$ such that $d(a^m, a^n) < \epsilon$ whenever $m, n > N(\epsilon)$. Notice that even though $a = \{a_r\} \notin X$, we may still apply the triangle inequality to d as long as all terms in the inequality arev finite: $d(a^m, a^n) \leq d(a^m, a) + d(a, a^n)$. From the construction of $a^m = \{a_r^m\}$ we obtain: $d(a^m, a) = \sum_{r=m+1}^{\infty} 2^{-r}\cdot(r - m)$. The latter sum can be shown to converge as follows:

$$\sum_{r=m+1}^{\infty} 2^{-r}(r - m) = 2^{-m} \sum_{r=m+1}^{\infty} 2^{-(r-m)}(r - m) \leq 2^{-m}\cdot \sum_{r=0}^{\infty} 2^{-r}\cdot r = 2^{-m}\cdot\frac{4}{3}$$

The last equality above follows from the fact that $\sum_{r=0}^{\infty} x^r = \frac{1}{1-x}$, whenever $|x| < 1$, and from the fact that $\frac{d}{dx}\left\{\frac{1}{1-x}\right\} = \frac{1}{(1-x)^2}$. We thus obtain: $d(a^m, a^n) \leq d(a^n, a) + d(a^m, a) < (2^{-m} + 2^{-n})\cdot\frac{4}{3}$. Clearly, the RHS tends to zero as $m, n \to \infty$. Thus, the sequence $a^m = \{a_r^m\}$ is Cauchy in X.

Section 6.4 Contraction

- Lecture 5 Definition 13: Let (X, d) be a nonempty complete metric space. An operator is a function $T : X \to X$.
- Lecture 5 Definition 13: An operator T is a contraction of modulus β if $\beta < 1$ and $\forall x, y \in X$ $d(T(x), T(y)) \leq \beta d(x, y)$.
- Lecture 5 Theorem 14: Every contraction is uniformly continuous.

- Lecture 5 Theorem 15 Contraction Mapping Theorem: Let (X, d) be a complete metric space, $T : X \to X$ a contraction with modulus $\beta < 1$. Then T has a unique fixed point $x*$. And for every $x_0 \in X$, the sequence defined by $x_1 = T(x_0)$, $x_2 = T(x_1) \ldots x_{n+1} = T(x_n)$ converges to $x*$.

Example 6.4.1
Show $T(x) = x - \frac{1}{x}$ on $(0, \infty)$ is not a contraction.
Solution:
$\forall x, y \in (0, \infty)$, $\frac{d(T(x),T(y))}{d(x,y)} = \frac{|s-y-(\frac{1}{x}-\frac{1}{y})|}{|x-y|} = 1 + \frac{1}{xy} > 1$. So we cannot find any $\beta < 1$ that qualifies as a contraction.

Example 6.4.2
Let $X = C([0, \beta], \beta < 1$. Let $d(f, g) = \{\max_t \big| f(t) - g(t) \big|\}$. Define $T : X \to X$ by $Tf(t) = \int_0^t f(s)ds$. Show that T has a unique fixed point.
Solution:
It suffices to show that T is a contraction:

$$\begin{aligned} d(Tf,\, Tg) &= max_t|Tf(t) - Tg(t)| \\ &= max_t|\int_0^t f(s)\, ds - \int_0^t g(s)\, ds| \\ &\leq max_t \int_0^t |f(s) - g(s)|\, ds \\ &\leq \int_0^\beta max_t\{|f(t) - g(t)|\}\, ds \\ &= \beta{\cdot}d(f,\, g) \end{aligned}$$