**Econ 204 Section 5**
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Key Words

Open Set, Closed Set, Interior, Closure, Exterior, Boundary, Limits of Function, Continuity, Uniform Continuity, Lipschitz Functioins, Homemorphism

**Section 5.1** Open Set and Closed Set

- Lecture 4 Definition 1: Let $(X, d)$ be a metric space. A set $A \subseteq X$ is **open** if $\forall x \in A \exists$ $\varepsilon > 0 \;\; B_\varepsilon(x) \subseteq A$. A set $C \subseteq X$ is **closed** if $X \setminus C$ is open.
- Lecture 4 Definition 3:
  - int$A$: the **interior** of $A$, the largest open set contained in $A$ (the union of all open sets contained in $A$)
  - $\overline{A}$: the **closure** of $A$, the smallest closed set containing $A$ (the intersection of all closed sets containing $A$)
  - ext$A$: the **exterior** of $A$, the largest open set contained in $X \setminus A$
  - $\partial A$: the **boundary** of $A$, $(X \setminus A) \bigcap A$
- Lecture 4 Theorem 2: Let $(X, d)$ be a metric space. The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open. The intersection of a finite collection of open sets is open.
- Lecture 4 Theorem 4: A set $A$ in a metric space $(X, d)$ is closed if and only if $\{x_n\} \subset A$, $\{x_n\} \to x \in X \Rightarrow x \in A$

  Openness and closedness depend on the underlying metric space as well as on the set.

  Theorem 2 is useful for proving that a set is open.

  Theorem 4 is useful for proving that a set is not closed. Unfortunately, using this theorem to prove that a set is closed is considerably more difficult, but still possible in many cases.

**Example 5.1.1** Is it true that the intersection of an infinite collection of open sets is open. Prove the statement or give a counterexample.

Solution:

False. Let $A_n = (-1, \frac{1}{n})$. $\cap_{n=1}^{\infty} A_n = (-1, 0]$ which is neither open nor closed. Notice that we can express a closed interval in $\mathbf{R}$ as the intersection of open intervals. $[a, b] = \cap_{n=1}^{\infty}(a - \frac{1}{n}, b + \frac{1}{n})$

**Example 5.1.2** Prove that the intersection of an arbitrary collection of closed and that the union of a finite collection of closed sets is closed.

Solution: From de Morgan's law we know that $\neg(A \bigwedge B) = \neg A \bigvee \neg B$. Then the result comes directly from Theorem 2 and the definition of closed set.

**Example 5.1.3** State whether the following sets are open, closed, both, or neither:

1. $\{(x, 0) \mid x \in (0, 1)\}$ in $\mathbf{R}^2$
2. $\{(x, y, z) \mid 0 \leq x + y \leq 1, z = 0\}$ in $\mathbf{R}^3$
3. $\{\frac{1}{n} \mid n \in \mathbf{N}\}$ in $\mathbf{R}$
4. $\{\frac{1}{n} \mid n \in \mathbf{N}\}$in $(0, \infty)$
5. $\mathbf{Q}$ in $\mathbf{R}$

Solution:

1. Neither 2. Closed 3. Neither 4. Closed 5. Neither

**Example 5.1.4** What is the closure of $\mathbf{Q}$ (the set of all rational numbers)

Solution:

$\overline{\mathbf{Q}} = \mathbf{R}$.

If $\forall a, b \in \mathbf{R}\, a < b \Rightarrow \overline{\mathbf{Q}} \cap [a, b] = [a, b]$, then $\overline{\mathbf{Q}} \cap \mathbf{R} = \cup_{\mathrm{n} \in \mathbf{Z}} \{\overline{\mathbf{Q}} \cap [n,\, n+1] = \cup_{\mathrm{n} \in \mathbf{Z}} [n.n+1] = \mathbf{R}$.So our job is to show $\overline{\mathbf{Q}} \cap [a, b] = [a, b]$. Consider the open interval $(a, b)$. By definition $\overline{\mathbf{Q}}$ is closed, hence $\overline{\mathbf{Q}}^c$ is open. So $(a, b) \cap \overline{\mathbf{Q}}^c$ is open. If $(a, b) \cap \overline{\mathbf{Q}}^c \neq \emptyset$, then there exists $x \in (a, b) \cap \overline{\mathbf{Q}}^c$ and $\varepsilon > 0$, such that $(x - \varepsilon,\, x + \varepsilon) \subseteq (a, b) \cap \overline{\mathbf{Q}}^c$. By the density of the rationals in $\mathbf{R}$ , there exists $q \in \mathbf{Q}$ such that $x - \varepsilon < q < x + \varepsilon$. Hence $q \in (a, b) \cap \overline{\mathbf{Q}}^c \subseteq \overline{\mathbf{Q}}^c$, contradiction. Hence $(a, b) \cap \overline{\mathbf{Q}}^c = \emptyset$. Then we have

$$(a, b) = \{(a, b) \cap \overline{\mathbf{Q}}\} \cup \{(a, b) \cap \overline{\mathbf{Q}}^c\} = \{(a, b) \cap \overline{\mathbf{Q}}\} \subseteq \overline{\mathbf{Q}}$$

Since $a$, $b$ are limit points of $(a, b)$ and $\overline{\mathbf{Q}}$ contains all of its limit points, it follows that $[a, b] \cap \overline{\mathbf{Q}} = [a, b]$.

**Example 5.1.5** Boundary/Interior/Closure

Find the boundary, interior, and closure of the following sets:
1. $\{(x, y) \in \mathbf{R}^2 \mid x^2 - y^2 > 5\}$.
2. $\{(x, y) \in \mathbf{R}^2 \mid x \geq y\}$.
3. $\{(x, y) \in \mathbf{R}^2 \mid x \in \mathbf{Q}\}$, where $\mathbf{Q}$ denotes the rational numbers.

Solution:

1. The boundary of this set is a hyperbola: $\{(x, y) \in \mathbf{R}^2 \mid x^2 - y^2 = 5\}$. The interior is the entire set: $\{(x, y) \in \mathbf{R}^2 \mid x^2 - y^2 > 5\}$. The closure is the union of the entire set and its boundary: $\{(x, y) \in \mathbf{R}^2 \mid x^2 - y^2 \geq 5\}$.
2. The boundary of this set is a diagonal line: $\{(x, y) \in \mathbf{R}^2 \mid x = y\}$. The interior is the entire set minus this diagonal boundary line: $\{(x, y) \in \mathbf{R}^2 \mid x > y\}$. The closure is the entire set: $\{(x, y) \in \mathbf{R}^2 \mid x \geq y\}$.
3. The boundary of this set is the entire plane: $\mathbf{R}^2$. The interior is the empty set: $\varnothing$. The closure is the entire plane: $\mathbf{R}^2$.

**Section 5.2** Continuity and Uniform Continuity

- Lecture 4 Definition 5: Let $(X, d)$ and $(Y, \rho)$ be metric spaces, $f : X \to Y$ . $f$ is **continuous at a point** $x_0 \in X$ if

  $$\forall \varepsilon > 0 \, \exists \, \delta(x_0, \varepsilon) > 0 \;\; d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x),\, f(x_0)) < \varepsilon$$

  f is continuous if it is continuous at every element of its domain.
- Lecture 4 Definition 8: Suppose $f : (X, d) \to (Y, \rho)$. $f$ is **uniformly continuous** if

  $$\forall \varepsilon > 0 \, \exists \, \delta(\varepsilon) > 0 \, \forall x_0 \in X \;\; d(x, x_0) < \delta(\varepsilon) \Rightarrow \rho(f(x),\, f(x_0)) < \varepsilon$$

- Lecture 4 Theorem 6: Let $(X, d)$ and $(Y, \rho)$ be metric spaces, $f : X \to Y$ . Then $f$ is continuous if and only if $\forall A \subseteq Y$ $A$ open in $Y \Rightarrow f^{-1}(A)$ is open in $X$.
- Lecture 4 Theorem 7: Let $(X, d_X)$, $(Y, d_Y)$ and $(Z, d_Z)$ be metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.

**Example 5.2.1** Uniformly Convergent

Let $f_n$ be a sequence of real-valued functions from $X$ to $\mathbf{R}$. We say that $f_n$ converges uniformly to $f$ if for every $\varepsilon > 0$, there exists an $n \in \mathbf{N}$ such that for all $x \in X$ and all $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$. Define $f_n : [0, 1] \to \mathbf{R}$ by the equation $f_n(x) = x^n$. Show that the sequence $f_n$ converges for each $x \in [0, 1]$, but not uniformly.

Solution:

To show pointwise convergence, we will show that $f_n \to 0$ for $x \in [0, 1)$ and that $f_n \to 1$ for $x = 1$. First, let $x \in [0, 1)$. Given $\varepsilon > 0$, choose $N(\varepsilon) > \log_x \varepsilon$. Then for $n \geq N(\varepsilon)$, $|f_n(x) - 0| = x^n \leq x^{N(\varepsilon)} < x^{\log_x \varepsilon} = \varepsilon$. Second, let $x = 1$. Then $x^n = 1$ for all $n$, so clearly $x^n \to 1$.

The sequence $f_n$ appears to be converging to the function

$$f(x) = \begin{cases} 0 & \text{if } x < 1 \\ 1 & \text{if } x = 1 \end{cases} .$$

Now we will show that this convergence is not uniform. We will show that there is no single $N(\varepsilon)$ that works for all $x \in [0,1]$. This is not surprising because the $N(\varepsilon)$ we used above depended on $x$. Let $\varepsilon < 1/3$. We will show that for any $N$, there is some $x$ such that $|x^N - f(x)| > \varepsilon$. Because $x^n$ is continuous for all $n$ we know that $x^N$ is continuous at $x = 1$. This means that there exists a $\delta > 0$ such that $|1-x| < \delta \Rightarrow |1-x^N| < \varepsilon$.
Now choose $x \in B_\delta(1) \cup [0,1)$. Then $|1-x^N| < \varepsilon < 1/2$ implies that $x^N > 1/2$. However, $|x^N - f(x)| = |x^N - 0| = x^N > 1/2 > \varepsilon$, so it is not true that $n \geq N \Rightarrow |x^N - f(x)| < \varepsilon$. We conclude that $f_n$ does not converge uniformly to $f$.

**Example 5.2.2** Let $f_n : X \to Y$ be a sequence of continuous functions. Let $x_n$ be a sequence of points of $X$ converging to $x$. Show that if the sequence $f_n$ converges uniformly to $f$, then $f_n(x_n)$ converges to $f(x)$.
Solution:
Our proof relies on the fact that $f$ must itself be continuous (which we must prove). Once we establish this fact, the rest of the proof is as follows:
Given an $\varepsilon > 0$, continuity implies that we can choose $\delta(\varepsilon) > 0$ such that $d(y,x) < \delta \Rightarrow |f(y) - f(x)| < \varepsilon$. Also, $\{x_n\} \to x$ means that we can choose $N_1(\delta)$ such that $n > N_1(\delta)$ implies that $d(x_n, x) < \delta$, so $|f(x_n) - f(x)| < \varepsilon$. Finally, by the uniform convergence of $\{f_n\}$, we can find $N_2(\varepsilon)$ such that $n > N_2(\varepsilon) \Rightarrow |f_n(x) - f(x)| < \varepsilon$ for all $x \in X$. Let $N = max\{N_1(\delta(\varepsilon/2)), N_2(\varepsilon/2)\}$. Then $n > N$ implies

$$|f_n(x_n) - f(x)| \leq |f_n(x_n) - f(x_n)| + |f(x_n) - f(x)| < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Now we must prove that $f$ is continuous, using the fact that $f_n$ is continuous for all $n$ and $\{f_n\}$ converges to $f$ uniformly. Given $\varepsilon > 0$, find $N(\varepsilon)$ such that $n > N(\varepsilon) \Rightarrow \forall_{y \in X} |f_n(y) - f(y)| < \varepsilon/3$. Let $n = N(\varepsilon) + 1$. Since $f_n$ is continuous, there exists $\delta(\varepsilon) > 0$ such that $d(y,x) < \delta(\varepsilon) \Rightarrow |f_n(y) - f_n(x)| < \varepsilon/3$. Then $d(y,x) < \delta$ implies

$$|f(y) - f(x)| \leq |f(y) - f_n(y)| + |f_n(y) - f_n(x)| + |f_n(x) - f(x)|,$$

which is less than $\varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon$.

**Example 5.2.3** Consider a subset A of an arbitrary metric space $(X, d)$. The distance between a point $x \in X$ and the set $A$ is defined as $d(x, A) = inf_{a \in A} d(x, a)$. Define the function $f : X \to \mathbf{R}$ by $f(x) = d(x, A)$, and let the metric on the range be Euclidean.
1. Show that $|f(x) - f(y)| \leq d(x,y) \, \forall x, y \in X$.
2. Show that $f$ is uniformly continuous.
Solution:
1. If $f(x) = f(y)$, then the inequality holds trivially. Without loss of generality, let $f(x) > f(y)$. If $f(y) = 0$, then $y$ is in the closure of $A$ so $\big|f(x) - f(y)\big| = f(x) = d(x, A) = \inf_{a \in A} d(x, a) \leq d(x, y)$. Finally, if $f(x) > f(y) > 0$ we have to do a bit more to prove the inequality. First, note that for all $\varepsilon > 0$, $\exists y_\varepsilon \in A$ such that $d(y, A) \leq d(y, y_\varepsilon) < d(y, A) + \varepsilon$. Furthermore, as $\varepsilon \to 0$, $d(y, y_\varepsilon) \to d(y, A)$ and observe that $d(x, y_\varepsilon) \geq d(x, A)$ for all $\varepsilon > 0$. Applying the triangle inequality, for all $\varepsilon > 0$ we have $d(x, y_\varepsilon) \leq d(x, y) + d(y, y_\varepsilon)$. It follows then that $d(x, A) - d(y, y_\varepsilon) \leq d(x, y_\varepsilon) - d(y, y_\varepsilon) \leq d(x, y)$ As $\varepsilon \to 0$, $d(x, A) - d(y, y_\varepsilon) \to d(x, A) - d(y, A)$. Hence $d(x, A) - d(y, A) \leq d(x, y)$. So we have $\big|f(x) - f(y)\big| = f(x) - f(y) = d(x, A) - d(y, A) \leq d(x, y)$.
2. Given $\varepsilon > 0$, take $\delta = \varepsilon$. Now for any $x, y \in X$, it follows that $d(x, y) < \delta \Rightarrow \big|f(x) - f(y)\big| \leq d(x, y) < \delta = \varepsilon$. Note that does not depend on $x$ or $y$ so the function is uniformly continuous.

**Section 5.3** Lipschitz Functioins and Homemorphism

- Lecture 4 Definition 9: Let $X, Y$ be normed vector space,$E \subseteq X$. $f : X \to Y$ is **lipschitz** on $E$ if

$$\exists K > 0 \, \forall x, z \in E \, \big\|f(x) - f(z)\big\|_Y \leq K \big\|x - z\big\|_X$$

$f$ is **locally Lipschitz** on $E$ if $\forall x_0 \in E\, \exists \varepsilon > 0$ $f$ is Lipschitz on $B_\varepsilon(x_0) \cap E$

Lipschitz $\Rightarrow$ locally Lipschitz $\Rightarrow$ continuous

Lipschitz $\Rightarrow$ uniformly continuous

- Lecture 4 Definition 10: Let $(X, d)$ and $(Y, \rho)$ be metric spaces. A function $f : X \to Y$ is called a homeomorphism if it is one-to-one and continuous, and its inverse function is continuous on $f(X)$.

**Example 5.3.1**

Prove whether the following functions are Lipschitz and whether they are locally Lipschitz.

1. $f(x) = x^2$ $x \in \mathbf{R}$

2. $f(x) = \sqrt{x}$ $x \in [0, \infty)$

Solution:

1. We prove that this function isn't uniformly continuous, therefore it's not Lipschitz.

WLOG, Choose $x_0 > 0$.Let $\varepsilon = 1 > 0$.Choose $x = x_0 + \frac{1}{x_0} \Rightarrow |\ f(x_0) - f(x)\ |= \frac{1}{{x_0}^2} + 2 > \varepsilon$. So $\delta(\varepsilon, x_0)$ must be chosen small enough so that $\delta(\varepsilon, x_0) < \frac{1}{x_0}$ which converges to zero as $x_0 \to \infty$. Hence $f(x)$ is not uniformly continuous.

As for locally Lipschitz, let $x \in B_\varepsilon(x)$. It turns out that it doesn't matter which $\varepsilon$ that we use; the slope of $f$ only gets arbitrarily large if we have an infinite neighborhood($\varepsilon \to \infty$), so let's just fix $\varepsilon$ as some positive number. Some reverse engineering shows that $K = 2(|x| + \varepsilon)$ will work. Let's check this. For $y, z \in B_\varepsilon(x)$, $((f(y)-f(z))/(y-z)) = ((|y^2-z^2|)/(|y-z|)) = ((|(y-z)(y+z)|)/(|y-z|)) = ((|y-z||y+z|)/(|y-z|)) = |y+z| < 2(|x| + \varepsilon) = K$.

Hence $f(x)$ is locally Lipschitz.

2. Lipschitz implies locally Lipschitz, so not locally Lipschitz implies not Lipschitz. Suppose $f$ is locally Lipschitz; then for the point $x_0 = 0$ we get an $\varepsilon$ and a $K$ from the definition. We will prove these can't work. Reverse engineer the pair $(y, 0) = (\min(\frac{\varepsilon}{2}, \frac{1}{4K^2}), 0)$ and you get $(y, 0) \in B_\varepsilon(0)$ but $((f(y)-f(0))/(y-0)) = \frac{\sqrt{y}}{y} = \frac{1}{\sqrt{y}} = (\min(\frac{\varepsilon}{2}, \frac{1}{4K^2}))^{-\frac{1}{2}} = \max(\frac{\varepsilon}{2}^{-\frac{1}{2}}, 2K) \geq 2K > K$.

Hence $f(x)$ is neither locally Lipschitz nor Lipschitz.