Econ 204 Section 3
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Key Words

Field, Vector Space, Upper Bound, Lower Bound, Supremum, Infimum, Completeness Axiom, Supremum Property, Intermediate Value Theorem.

Section 3.1 Field and Vector Space

- Lecture 2 Section 1.5 Field Axioms (nine properties).
- Lecture 2 Section 1.5 Vector Space Axioms (eight properties).

Example 3.1.1 Show that $\mathbf{Q}(\sqrt{2})$ over $\mathbf{Q}$ is a vector space.
Recall in lecture $\mathbf{Q}(\sqrt{2}) = \{q + r\sqrt{2} : q, r \in \mathbf{Q}\}$. Check the eight properties.
Associativity of $+$:

$$\forall q_1+r_1\sqrt{2}, q_2+r_2\sqrt{2}, q_3+r_3\sqrt{2} \in \mathbf{Q}(\sqrt{2}),\ (q_1+r_1\sqrt{2}+q_2+r_2\sqrt{2})+q_3+r_3\sqrt{2} = q_1+r_1\sqrt{2}+(q_2+r_2\sqrt{2}+q_3+r_3\sqrt{2})$$

Commutativity of $+$:

$$\forall q_1 + r_1\sqrt{2},\ q_2 + r_2\sqrt{2} \in \mathbf{Q}(\sqrt{2}),\ q_1 + r_1\sqrt{2} + q_2 + r_2\sqrt{2} = q_2 + r_2\sqrt{2} + q_1 + r_1\sqrt{2}$$

Existence of vector additive identity:

$$\exists 0 = 0 + 0\sqrt{2} \in \mathbf{Q}(\sqrt{2}),\ \forall q_1 + r_1\sqrt{2} \in \mathbf{Q}(\sqrt{2}),\ q_1 + r_1\sqrt{2} + 0 = q_1 + r_1\sqrt{2}$$

Existence of vector additive inverse:

$$\forall q + r\sqrt{2} \in \mathbf{Q}(\sqrt{2}),\ \exists(-q) + (-r)\sqrt{2} \in \mathbf{Q}(\sqrt{2}),\ s.t.\ q + r\sqrt{2} + (-q + (-r)\sqrt{2}) = 0$$

Distributivity of scalar multiplication over vector addition:

$$\forall a \in \mathbf{Q}, \forall q_1+r_1\sqrt{2},\ q_2+r_2\sqrt{2} \in \mathbf{Q}(\sqrt{2}),\ a\cdot(q_1+r_1\sqrt{2}+q_2+r_2\sqrt{2}) = a\cdot(q_1+r_1\sqrt{2})+a\cdot(q_2+r_2\sqrt{2})$$

Distributivity of scalar multiplication over scalar addition:

$$\forall a, b \in \mathbf{Q}, \forall q + r\sqrt{2} \in \mathbf{Q}(\sqrt{2}),\ (a + b) \cdot (q + r\sqrt{2}) = a \cdot (q + r\sqrt{2}) + b \cdot (q + r\sqrt{2})$$

Associativity of $\cdot$:

$$\forall a, b \in \mathbf{Q}, \forall q + r\sqrt{2} \in \mathbf{Q}(\sqrt{2}),\ (a \cdot b) \cdot (q + r\sqrt{2}) = a \cdot \{b \cdot (q + r\sqrt{2})\}$$

Multiplicative identity:

$$\forall q + r\sqrt{2} \in \mathbf{Q}(\sqrt{2}),\ \exists 1 \in \mathbf{Q},\ 1 \cdot (q + r\sqrt{2}) = q + r\sqrt{2}$$

Pay attention to the differences between a vector space and a field.

Section 3.2 Supremum and Infimum

- Lecture 2 Definition 2: Suppose $X \subseteq \mathbf{R}$. We say u is an **upper bound** for X if $\forall x \in X\ \ x \leq u$ and l is a **lower bound** for X if $\forall x \in X\ \ x \geq l$.
- Lecture 2 Definition 3: Suppose X is bounded above. The **supremum** of X, written $\sup X$, is the smallest **upper bound** for X, i.e. $\sup X$ satisfies $\forall x \in X\ \ x \leq \sup X$ and $\forall y < \sup X$ $\exists x \in X\ \ x > y$. The **infimum** of X, written $\inf X$, is the largest **lower bound** for X, i.e. $\inf X$ satisfies $\forall x \in X\ \ x \geq \inf X$ and $\forall y > \inf X\ \ \exists x \in X\ \ x < y$.

Example 3.2.1 Sup and Inf
Suppose $X = \{\frac{1}{x} \mid 1 \leq x < +\infty\}$. Use the definition in the lecture to show that $\sup X = 1$ and $\inf X = 0$.
Solution:

For any $\frac{1}{x} \in X$, $1 \leq x \Rightarrow \frac{1}{x} \leq 1$. So 1 is an upper bound for X. $\forall y < 1\, \exists x = 1 \in X\; s.t.\; x > y$. So 1 is the smallest upper bound for X. Hence $\sup X = 1$.

Similarly, for any $\frac{1}{x} \in X$, $1 \leq x \Rightarrow \frac{1}{x} > 0$. So 0 is a lower bound for X. $\forall 0 < y \leq 1$, $\exists \frac{1}{x_y} = \frac{y}{2} \in X$, $\frac{1}{x_y} < y$. $\forall y > 1$, $\exists 1 \in X$, $1 < y$. So 0 is the largest lower bound for X. Hence $\inf X = 0$.

Remember the two steps. First, show it's an upper bound (lower bound). Second, show it's the smallest upper bound (largest lower bound).

Example 3.2.2 Alternative Definition of Sup

Suppose $X \subseteq \mathbf{R}$ is bounded above. Prove that $a \in \mathbf{R}$ is a supremum for X if and only if that a is an upper bound for X and for all $\varepsilon > 0$ there exists some $x \in X$ such that $a - x < \varepsilon$.

Solution:

To prove if and only if statement, we have to show it is true in both directions.

$\Rightarrow$ We prove it by contradiction. Suppose that $\sup X = a$. By definition, x is an upper bound for X. Let $\varepsilon > 0$ and suppose that for all $x \in X$, $a - x \geq \varepsilon \Rightarrow a - \varepsilon \geq x$. This implies that $a - \varepsilon$ is an upper bound for X. Since $a - \varepsilon < a$, a contradiction of the fact that a is the smallest upper bound for X. So for all $\varepsilon > 0$ there exists some $x \in X$ such that $a - x < \varepsilon$.

$\Leftarrow$ We prove it by contradiction. Suppose that a is an upper bound for X and for all $\varepsilon > 0$ there exists some $x \in X$ such that $a - x < \varepsilon$. Let b be an upper bound for X. To prove that $\sup X = a$ we only have to show that $a \leq b$. Suppose not. $a > b \Rightarrow$ we can choose $\varepsilon = a - b > 0$ then there exists some $x \in X$ such that $a - x < \varepsilon = a - b$. This implies $b < x$ which is a contradiction of the fact that b is an upper bound for X. Hence $a \leq b$. Then it follows that $\sup X = a$.

We can write a similar definition of Inf:

Suppose $X \subseteq \mathbf{R}$ is bounded below. Prove that $b \in \mathbf{R}$ is a infimum for X if and only if that b is an lower bound for X and for all $\varepsilon > 0$ there exists some $x \in X$ such that $x - b < \varepsilon$.

This alternative definition of Sup and Inf is very useful for solving problems.

Example 3.2.3 Do you really understand Sups and Infs?

Suppose $A, B \subset \mathbf{R}$. A, B are bounded non-empty sets for the following questions.

1. The best we can say about $\sup(A \cup B)$ **is that**

(a) It is the maximum of $\sup(A)$ and $\sup(B)$.
(b) It is equal to at least one of $\sup(A)$ and $\sup(B)$.
(c) It is greater than or equal to $\sup(A)$ and greater than or equal to $\sup(B)$.
(d) It is less than or equal to $\sup(A)$ and less than or equal to $\sup(B)$.

Solution:

(a) is the best answer.

We prove by showing $\sup(A \cup B) \geq \max\{\sup(A), \sup(B)\}$ and $\sup(A \cup B) \leq \max\{\sup(A), \sup(B)\}$. First, since $A, B \subset A \cup B$, which implies that $\sup(A \cup B) \geq \sup(A)$ and $\sup(A \cup B) \geq \sup(B)$. Hence $\sup(A \cup B) \geq \max\{\sup(A), \sup(B)\}$. Since both A and B are bounded. By definition, $\sup(A) \geq x$ for any $x \in A$ and $\sup(B) \geq x$ for any $x \in B$, and thus $\max\{\sup(A), \sup(B)\} \geq x$ for any $x \in A$ or $x \in B$. Of course, this implies that $\max\{\sup(A), \sup(B)\} \geq x$ for any $x \in A \cup B$, which guarantees that $\sup(A \cup B) \leq \max\{\sup(A), \sup(B)\}$.

2. The best we can say about $\sup(A \cap B)$ **is that:**

(a) It is the minimum of $\sup(A)$ and $\sup(B)$.
(b) It is equal to at least one of $\sup(A)$ and $\sup(B)$.
(c) It is greater than or equal to at least one of $\sup(A)$ and $\sup(B)$.
(d) It is less than or equal to $\sup(A)$ and less than or equal to $\sup(B)$.

Solution:

Only (d) is correct.

$A \cap B \subset A$ and $A \cap B \subset B$, which implies that $\sup(A \cap B) \leq \sup(A)$ and $\sup(A \cap B) \leq \sup(B)$. It follows that $\sup(A \cap B) \leq \min\{\sup(A), \sup(B)\}$.

What's the counter example of b-d?

Let $A = \{1, 2, 3\}$ $B = \{1, 2, 4, 6\} \Rightarrow A \cap B = \{1, 2\}$. $\sup(A \cap B) = 2$, $\sup(A) = 3$, $\sup(B) = 6$

3. Which of the following statements would be equivalent to saying that $\inf A \leq \inf B$**?**

(a) There exists $a \in A$ and $b \in B$ such that $a < b$.

(b) For every $a \in A$ and every $b \in B$, we have $a \leq b$.

(c) For every $a \in A$ there exists $b \in B$ such that $a \leq b$.

(d) For every $b \in B$ there exists $a \in A$ such that $a \leq b$.

(e) For every $b \in B$ and $\varepsilon > 0$ there exists $a \in A$ such that $a < b + \varepsilon$.

Solution:

Only e is correct. Prove it by yourself. For a-d we can find counter examples.

4. Which of the following statements would be equivalent to saying that $\inf A \leq \sup B$**?**

(a) For every $a \in A$ and every $b \in B$, we have $a \leq b$.

(b) For every $a \in A$ there exists $b \in B$ such that $a \leq b$.

(c) There exists $b \in B$ such that $a \leq b$. for all $a \in A$.

(d) For every $b \in B$ there exists $a \in A$ such that $a \leq b$.

(e) For every $\varepsilon > 0$ there exists $a \in A$ and $b \in B$ such that $a < b + \varepsilon$.

Solution:

Only e is correct. Prove it by yourself. For a-d we can find counter examples.

5. Which of the following statements would be equivalent to saying that $\sup A \leq \inf B$**?**

(a) For every $a \in A$ and every $b \in B$, we have $a \leq b$.

(b) For every $a \in A$ there exists $b \in B$ such that $a \leq b$.

(c) There exists $b \in B$ such that $a \leq b$. for all $a \in A$.

(d) For every $b \in B$ there exists $a \in A$ such that $a \leq b$.

(e) For every $\varepsilon > 0$ there exists $a \in A$ and $b \in B$ such that $a < b + \varepsilon$.

Solution:

Only a is correct. Prove it by yourself. For b-e we can find counter examples.

6. Which of the following statements would be equivalent to saying that $\sup A \leq \sup B$**?**

(a) For every $a \in A$ and every $b \in B$, we have $a \leq b$.

(b) For every $a \in A$ there exists $b \in B$ such that $a \leq b$.

(c) There exists $b \in B$ such that $a \leq b$. for all $a \in A$.

(d) For every $a \in A$ and every $\varepsilon > 0$, there exists $b \in B$ such that $a < b + \varepsilon$.

(e) For every $\varepsilon > 0$ there exists $a \in A$ and $b \in B$ such that $a < b + \varepsilon$.

Solution:

Only d is correct. Prove it by yourself. For a-c and e we can find counter examples.

Section 3.3 Completeness Axiom and Supremum Property

- Lecture 2 Completeness Axiom
- Lecture 2 The Supremum Property
- Lecture 2 Theorem 4: The Supremum Property and the Completeness Axiom are equivalent.
  Completeness Axiom and The Supremum Property are very important. But the proof is not hard to understand. Study the lecture notes carefully and finish exercise 5 of problem set 1 by yourself.

Section 3.4 Intermediate Value Theorem

- Lecture 2 Theorem 5 Intermediate Value Theorem

  Example 3.4.1 A Special Case of Imtermediate Value Theorem
  Suppose $f(x)$ is a continous function on $[a,b]$ and $f(a)\cdot f(b)<0$. Use supremum property to show that there exists $c\in(a,b)$ such that $f(c)=0$.
  Solution:
  Without loss of generality, suppose $f(a)<0$, $f(b)>0$.
  Let $B=\{x\in[a,b]:\ f(x)<0\}$. $a\in B$, so $B\neq\emptyset$. $B\subseteq[a,b]$ so B is bounded above. By the Supremum Property, sup$B$ exists and is real. So let $c=$sup$B$.
  Since $a\in B$, $c\geq a$. Since $B\subseteq[a,b]$, $c\leq b$. Therefore, $c\in[a,b]$.
  We claim that $f(c)=0$.
  If not, suppose $f(c)<0$. Since $f(b)>0$, $c\neq b$, so $c<b$. Let $\varepsilon=-\frac{f(c)}{2}>0$. Since f is continuous at c, there exists $\delta>0$ such that $\left|x-c\right|<\delta\Rightarrow\left|f(x)-f(c)\right|<\varepsilon\Rightarrow f(x)<f(c)+\epsilon=f(c)-\frac{f(c)}{2}=\frac{f(c)}{2}<0$. So $(c,c+\delta)\subseteq B$. $c\neq$sup$B$, contradiction.
  Suppose $f(c)>0$. Then since $f(a)<0$, $c\neq a$, so $c>a$. Let $\varepsilon=\frac{f(c)}{2}>0$. Since f is continuous at c, there exists $\delta>0$ such that $\left|x-c\right|<\delta\Rightarrow\left|f(x)-f(c)\right|<\varepsilon\Rightarrow f(x)>f(c)-\epsilon=f(c)-\frac{f(c)}{2}=\frac{f(c)}{2}>0$. So $(c-\delta,c+\delta)\cap B=\emptyset$. So either there exists $x\in B$ with $x\geq c+\delta$ or $c-\delta$ is an upper bound for B. So $c\neq$sup$B$, contradiction.
  The structure of the proof is similar to the proof of lecture notes.