Econ 204
Problem Set 6 Solutions

Exercise 1

We know that there are solutions to the system $0 = F(x,y,w,z) = f(x,y) - (w,z)$. $DF(x,y,w,z) = Df(x,y) - D(w,z) = \begin{pmatrix} D_x f_1(x,y) & D_y f_1(x,y) & -1 & 0 \\ D_x f_2(x,y) & D_y f_2(x,y) & 0 & -1 \end{pmatrix}$, where $f_i(x,y)$ is the i^{th} component of f. DF is clearly of rank 2 for all (x,y,w,z) and since $f \in C^3$ it follow that $F \in C^3$. By Transversality Theorem we know that there is a subset B of R^2 such that B^c is of measure zero, and for all (x,y,w,z) satisfying $F(x,y,w,z) = 0$ where $(w,z) \in B$, we have $rank(D_{(x,y)}F(x,y,w,z)) = 2$. Hence, for these (x,y,w,z) we satisfy the hypotheses of the Implicit Function Theorem and we can find the desired implicit functions.

Exercise 2

Check directly. Since $0 \neq 1/2$, $x = 0$ cannot be a fixed point. Hence the only chance we have is if $x = 1/(x+1)$ if and only if $x(x+1) = 1$ if and only if $x_{1,2} = \frac{-1 \pm \sqrt{1+4}}{2}$. Thus f has one fixed point in the interval $[0,1]$. f is a set function (correspondence) so Brower's Thm. does not apply. Kakutani's does not apply since f is not convex valued.

For the next function, $x = \epsilon$ is a fixed point. You can also verify that the assumptions of Kakutani's Theorem are satisfied: $[0,1]$ is compact (closed and bounded), convex (an interval), and non-empty, and the correspondence f is: convex valued since image of every point is an interval; closed valued since intervals in the image are also closed; non-empty valued because we defined it that way; upper-hemicontinuous since f has a closed graph and the set $[0,1]$ is compact (Theorem 12 Lecture 7)

Exercise 3 .

Pick y_1 and y_2 in B_i,, then $y_1 = z_1 - x$ and $y_2 = z_2 - x$ where $z_1 R_i x$ and $z_2 R_i x$. By convexity of the relation R_i, $(\alpha z_1 + (1-\alpha) z_2) R_i x$ for all $\alpha \in (0,1)$. Thus $\alpha y_1 + (1-\alpha) y_2 = \alpha(z_1 - x) + (1-\alpha)(z_2 - x) = (\alpha z_1 + (1-\alpha) z_2) - x$, and since $(\alpha z_1 + (1-\alpha) z_2) R_i x$ we have $\alpha y_1 + (1-\alpha) y_2 \in B_i$. This holds for all $\alpha \in (0,1)$ and all $i = 1, ...m$, so B_i is convex for all $i = 1, ...m$.

Now Let $y_1, y_2 \in B$. Thus $y_1 = \sum_{i=1}^m (y_{i1} - x)$ and $y_2 = \sum_{i=1}^m (y_{i2} - x)$.

$\alpha y_1 + (1-\alpha) y_2 = \alpha \sum_{i=1}^m (y_{i1} - x) + (1-\alpha) \sum_{i=1}^m (y_{i2} - x) = \sum_{i=1}^m \{\alpha(y_{i1} - x) + (1-\alpha)(y_{i2} - x)\}$ where $\{\alpha(y_{i1} - x) + (1-\alpha)(y_{i2} - x)\} \in B_i$ by convexity of B_i and thus $\sum_{i=1}^m \{\alpha(y_{i1} - x) + (1-\alpha)(y_{i2} - x)\} \in B$ by definition.

If $0 \notin B$ as given, convexity of B and $\{0\}$ is all we need to apply the Separating Hyperplane Theorem and get some $p \neq 0$, $p \in \mathbf{R}^n$ such that $0 = p \cdot 0 = \sup p \cdot 0 \leq \inf p \cdot B$.

Exercise 4

Since $B \subset S_i$ for all $i \in I$, it follows that $B \subset \cap_{i \in 1} S_i$. Left to show the other set containment.

We know that for any two sets C, D we have $C \subset D$ if and only if $D^c \subset C^c$. Hence suppose that $x \in B^c$. This means that x is not an element of B. Since B is convex, we can apply the Separating Hyperplane Theorem and get $p \neq 0$, $p \in \mathbf{R}^n$ such that $p \cdot x \leq \inf p \cdot B < p \cdot y$ for all $y \in B$. How do we know that the infimum is not attained by any $y \in B$?:

Lemma:

I used the result that since B is open, $\inf p \cdot B$ is not attained by any element in B : for a contradiction, suppose that there was $y \in B$, such that $\inf p \cdot B = p \cdot y$; without loss of generality, take $p_i > 0$ (proof is almost the same if $p_i < 0$). Then since B is open, there is an open ball U around y such that $U \subset B$. But then there must exist an $\epsilon > 0$ such that $z = y - (0, ..., \epsilon, ...0) \in U$ where ϵ is in the $i'th$ entry. Hence, $z \in B$, but $p \cdot z < p \cdot y$. Thus infimum is not attained by any $y \in B$.

Now let $S_j = \{y \in \mathbf{R}^n : p \cdot y \leq \inf p \cdot B\}$. S_j contains x and S_j^c is an open half-space containing B that does not contain x. Thus x cannot be in the intersection of all open half-spaces containing B. Hence we've shown that if $x \in B^c$ then $x \in (\cap S_{i \in I})^c$.

Exercise 5

Suppse $f(x) = \ln x$ is Lipschitz, then there exists a constant $K \in \mathbf{R}$ such that $|\ln x - \ln y| \leq K |x - y|$ for $x, y > 0$. Consider the mean value expansion for a differentiable function $f : \mathbf{X} \to \mathbf{R}$ where X is open and convex**:** $f(y) - f(x) = f'(z)(y - x)$ for some $z \in (x, y)$. Note that $\frac{d}{dx}(\ln x) = \frac{1}{x}$ is unbounded on $(0, \infty)$. Hence, if $0 < x_0 < \frac{1}{K}$ we will have $K < \frac{1}{x_0} = \frac{d}{dx}(\ln x)|_{x=x_0}$ and since the derivative of $\ln x$ is strictly decreasing, for every $y < x_0$, by the mean value expansion we have $f(y) - f(x) > K(y - x)$ with $x < y < x_0$, which is a contradiction. $f(x) = \ln x$ however is Lipschitz on any set of the form $[a, \infty)$ where $a > 0$ since it's derivative is bounded (see part (c)).

On the other hand $\left|\frac{d}{dx}\cos(x)\right| = |\sin x| \leq 1$, so by mean value expansion $|\cos x - \cos y| \leq |x - y|$, so (b) is Lipschitz. Finally for (c) we have $|f(y) - f(x)| = |f'(z)(y - x)| \leq M |y - x|$ so it's Lipschitz as well.

The differential equation $\frac{dy}{dt} = \frac{3}{2} y(t)^{1/3}$ with $y(t_0) = 0$ has a solution since the function on the right is continuous (Theorem 2, Lecture 14). Since the function is not Lipschitz (derivative unbounded, the solution may not be unique). Check that the $\frac{d}{dy} C y^{1/3}$ is unbounded near 0. Also check that $y(t) = t^{3/2}$ is a solution. Let $y_\theta(t) = (t - \theta)^{3/2}$ if $t \geq \theta > 0$ and $y(t) = 0$ otherwise. This is also a solution: check lecture 14 for an identical proof.

Exercise 6

a) We get the parabola $y = x^2$ and the lines $y = 1$ and $y = 0$. The steady state with both $x, y > 0$ is the point $(1, 1)$.

b) When we linearize the system around $(1, 1)$, we get

$$\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} (x(t)-1)' \\ (y(t)-1)' \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x(t)-1 \\ y(t)-1 \end{pmatrix}$$

Letting $z(t) = x(t) - 1$ and $w(t) = y(t) - 1$ we can write the system as:

$$\begin{aligned} z'(t) &= 2 \times z(t) - w(t) \\ w'(t) &= w(t) \end{aligned}$$

c) Eigenvalues of the matrix $\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$ are 2 and 1 with a pair of eigenvectors $(1, 0)$ and $(1, 1)$: $A \times (1, 0) = (2, 0) = 2 \times (1, 0)$ and $A \times (1, 1) = (1, 1) = 1 \times (1, 1)$. Both eigenvalues are positive, so the solutions diverge to infinity.

The general solution of the system is of the form:

$$\begin{aligned} z(t) &= C_{11} e^{2(t-t_0)} + C_{12} e^{(t-t_0)} \\ w(t) &= C_{21} e^{2(t-t_0)} + C_{22} e^{(t-t_0)} \end{aligned}$$

If we transform the system into the basis formed by the eigenvectors of $\begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix}$ by setting $\begin{pmatrix} h(t) \\ k(t) \end{pmatrix} = U^{-1} \begin{pmatrix} z(t) \\ w(t) \end{pmatrix}$, where U^{-1} is the change of basis matrix from the standard basis to the basis formed by the two eigenvectors, then we can rewrite the system as:

$$\begin{pmatrix} h'(t) \\ k'(t) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h(t) \\ k(t) \end{pmatrix}$$

and get the solution $h(t) = Ke^{2(t-t_0)}$, $k(t) = Me^{(t-t_0)}$. $\begin{pmatrix} z(t) \\ w(t) \end{pmatrix} = U \begin{pmatrix} h(t) \\ k(t) \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} h(t) \\ k(t) \end{pmatrix} = \begin{pmatrix} Ke^{2(t-t_0)} + Me^{(t-t_0)} \\ Me^{(t-t_0)} \end{pmatrix}$, where $K = C_{11}$, $M = C_{12} = C_{22}$. C_{21} must then be zero.