Economics 204–First Midterm Test–August 30, 2004, 6-9pm
Each question is worth 20% of the total
Please use separate bluebooks for Parts I and II

Part I

1. State and prove a theorem on the uniqueness of limits of sequences in metric spaces.

2. Consider the function

$$f(x, y) = x^3 + y^3 + 2x^2 - 2xy - y^2 - 3x - 6y$$

   (a) Compute the first order conditions for a local maximum or minimum of f. Verify these are satisfied at the point $(x_0, y_0) = (1, 2)$.

   (b) Compute $D^2 f(x_0, y_0)$ and give the quadratic Taylor series for f at the point (x_0, y_0).

   (c) Find the eigenvalues of $D^2 f(x_0, y_0)$ and determine whether f has a local max, a local min, or a saddle at (x_0, y_0).

   (d) Find an orthonormal basis for $\mathbf{R}^2$ consisting of eigenvectors $D^2 f(x_0, y_0)$. Rewrite the quadratic Taylor series for f at the point (x_0, y_0) in terms of this basis.

   (e) Use the Taylor series found in part (d) to describe the approximate shape of the level sets of f near the point (x_0, y_0).

3. Prove that if a set X has n elements, then 2^X, the set of all subsets of X, has 2^n elements. Hint: use induction.

Part II

4. Suppose X, Y, Z are finite-dimensional vector spaces over $\mathbf{R}$ with bases U, V, W respectively, $S \in L(X, Y)$ and $T \in L(Y, Z)$. Summarize the relationships among S, T, $T \circ S$, and their matrix representations using a commutative diagram.[1] Explain the interpretation of the diagram.

5. Consider the metric space (X, d), where $X = \mathbf{Q} \cap [\mathbf{0}, \mathbf{1}]$, $\mathbf{Q}$ is the set of all rational numbers, and d is the usual Euclidean metric $d(x, y) = |x - y|$. Show that (X, d) is *not* compact by exhibiting an open cover of X that has no finite subcover.

[1] If you don't remember the commutative diagram given in class and the handout, don't panic. Think through the relationships and explain them, if possible with a diagram.