Economics 204

Lecture 5–Friday, July 31, 2009

Section 2.6 (Continued)

Properties of Real Functions

Theorem 1 (6.23, Extreme Value Theorem) *Let f be a continuous real-valued function on $[a, b]$. Then f assumes its minimum and maximum on $[a, b]$. In particular, f is bounded above and below.*

Proof: Let

$$M = \sup\{f(t) : t \in [a, b]\}$$

If M is finite, for each n, we may choose $t_n \in [a, b]$ such that $M \geq f(t_n) \geq M - \frac{1}{n}$ (if we couldn't make such a choice, then $M - \frac{1}{n}$ would be an upper bound and M would not be the supremum). If M is infinite, choose t_n such that $f(t_n) \geq n$. By the Bolzano-Weierstrass Theorem, $\{t_n\}$ contains a convergent subsequence $\{t_{n_k}\}$, with

$$\lim_{k\to\infty} t_{n_k} = t_0 \in [a, b]$$

Since f is continuous,

$$\begin{aligned} f(t_0) &= \lim_{t\to t_0} f(t) \\ &= \lim_{k\to\infty} f(t_{n_k}) \\ &= M \end{aligned}$$

so M is finite and

$$f(t_0) = M = \sup\{f(t) : t \in [a, b]\}$$

so f attains its maximum and is bounded above. The argument for the minimum is similar. ■

Theorem 2 (6.24, Intermediate Value Theorem) *Suppose* $f : [a, b] \to \mathbf{R}$ *is continuous, and* $f(a) < d < f(b)$. *Then there exists* $c \in (a, b)$ *such that* $f(c) = d$.

Proof: We did a hands-on proof already. Now, we can simplify it a bit. Let

$$B = \{t \in [a, b] : f(t) < d\}$$

$a \in B$, so $B \neq \emptyset$. By the Supremum Property, $\sup B$ exists and is real so let $c = \sup B$. Since $a \in B$, $c \geq a$. $B \subseteq [a, b]$, so $c \leq b$. Therefore, $c \in [a, b]$. We claim that $f(c) = d$.

Let

$$t_n = \min\left\{c + \frac{1}{n}, b\right\} \geq c$$

Either $t_n > c$, in which case $t_n \notin B$, or $t_n = c$, in which case $t_n = b$ so $f(t_n) > d$, so again $t_n \notin B$; in either case, $f(t_n) \geq d$. Since f is continuous at c, $f(c) = \lim_{n\to\infty} f(t_n) \geq d$ (Theorem 3.5 in de la Fuente).

Since $c = \sup B$, we may find $s_n \in B$ such that

$$c \geq s_n \geq c - \frac{1}{n}$$

Since $s_n \in B$, $f(s_n) < d$. Since f is continuous at c, $f(c) = \lim_{n\to\infty} f(s_n) \leq d$ (Theorem 3.5 in de la Fuente).

Since $d \leq f(c) \leq d$, $f(c) = d$. Since $f(a) < d$ and $f(b) > d$, $a \neq c \neq b$, so $c \in (a, b)$.■

Monotonic Functions:

Definition 3 A function f is *monotonically increasing* if

$$y > x \Rightarrow f(y) \geq f(x)$$

Theorem 4 (6.27) *Suppose f is monotonically increasing on (a, b). Then the one-sided limits*

$$f(t^+) = \lim_{u \to t^+} f(u)$$
$$f(t^-) = \lim_{u \to t^-} f(u)$$

exist and are real numbers for all $t \in (a, b)$.

Proof: This is analogous to the proof that a bounded monotone sequence converges. ■

(*We say that f has a simple jump discontinuity at t if the one-sided limits $f(t-)$ and $f(t+)$ both exist. The previous theorem says that monotonic functions have only simple jump discontinuities; note that monotonicity implies that $f(t-) \le f(t) \le f(t+)$.*)

Theorem 5 (6.28) *Suppose that f is monotonically increasing on (a, b). Then*

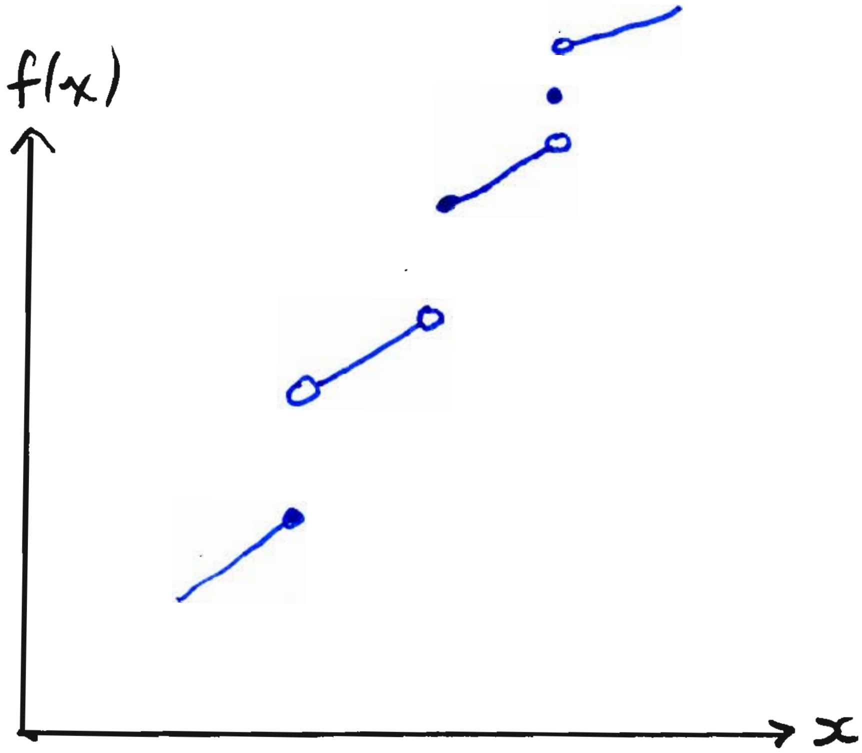
$$D = \{t : f \text{ is discontinuous at } t\}$$

is finite (possibly empty) or countable. ("A monotonic function is continuous almost everywhere.")

Proof: If $t \in D$, we have $f(t^-) < f(t^+)$ (if the left- and right-hand limits agreed, then by monotonicity they would have to equal $f(t)$, so f would be continuous at t). So for every $t \in D$, since $\mathbf{Q}$ is dense, we may choose

$$r(t) \in \mathbf{Q}, \;\; f(t^-) < r(t) < f(t^+)$$

This defines a function $r : D \to \mathbf{Q}$ (for those who care about these things, we have used the Axiom of Choice, which says that



Simple Jump Discontinuities

if we can choose such a rational r for each $t \in D$, then we can can choose a *function* $r : D \to \mathbf{Q}$). Notice that

$$s > t \Rightarrow f(s^-) \geq f(t^+)$$

so

$$s > t, s, t \in D \Rightarrow r(s) > f(s^-) \geq f(t^+) > r(t)$$

so $r(s) \neq r(t)$. Therefore, r is one-to-one, so it is a bijection from D to a subset of $\mathbf{Q}$, so D is finite or countable.■

Section 2.7: Complete Metric Spaces, Contraction Mapping Theorem

Roughly, a metric space is complete if "every sequence that ought to converge to a limit has a limit to converge to."

$x_n \to x$ means

$$\forall_{\varepsilon>0} \exists_{N(\varepsilon/2)}\ n > N(\varepsilon/2) \Rightarrow d(x_n, x) < \frac{\varepsilon}{2}$$

Observe that if $n, m > N(\varepsilon/2)$, then

$$d(x_n, x_m) \leq d(x_n, x) + d(x, x_m) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This motivates the following definition:

Definition 6 A sequence $\{x_n\}$ in a metric space (X, d) is *Cauchy* if

$$\forall_{\varepsilon>0} \exists_{N(\varepsilon)}\ n, m > N(\varepsilon) \Rightarrow d(x_n, x_m) < \varepsilon$$

(*A Cauchy sequence is trying really hard to converge, but there may not be anything for it to converge to.*)

Theorem 7 (7.2) *Every convergent sequence in a metric space is Cauchy.*

Proof: We just did it.■

Example: Let $X = (0,1]$, d the Euclidean metric. Let $x_n = \frac{1}{n}$. Then $x_n \to 0$ in E^1, so $\{x_n\}$ is Cauchy in E^1. But the Cauchy property depends only on the sequence and the metric d, not on the ambient metric space. So $\{x_n\}$ is Cauchy in (X,d), but $\{x_n\}$ does not *converge* in (X,d) because the point it is trying to converge to (0) is not an element of X.

Definition 8 A metric space (X,d) is *complete* if every Cauchy sequence $\{x_n\} \subseteq X$ converges to a limit $x \in X$. A *Banach space* is a normed space which is complete in the metric generated by its norm.

Example: Consider the earlier example of $X = (0,1]$, d the usual Euclidean metric. Since $x_n = \frac{1}{n}$ is Cauchy but does not converge, $((0,1],d)$ is not complete.

Example: $\mathbf{Q}$ is not complete in the Euclidean metric. To see this, let

$$x_n = \frac{\lfloor 10^n \sqrt{2} \rfloor}{10^n}$$

where as before, $\lfloor y \rfloor$ is the greatest integer less than or equal to y; x_n is just equal to the decimal expansion of $\sqrt{2}$ to n digits past the decimal point. Clearly, x_n is rational. $|x_n - \sqrt{2}| \leq 10^{-n}$, so $x_n \to \sqrt{2}$ in E^1, so $\{x_n\}$ is Cauchy in E^1, hence Cauchy in $\mathbf{Q}$; since $\sqrt{2} \notin \mathbf{Q}$, $\{x_n\}$ is not convergent in $\mathbf{Q}$, so $\mathbf{Q}$ is not complete.

Theorem 9 (7.10) $\mathbf{R}$ *is complete with the usual metric (so E^1 is a Banach space).*

Proof: Our proof is different from the one in de la Fuente. Suppose $\{x_n\}$ is a Cauchy sequence in $\mathbf{R}$. Fix $\varepsilon > 0$.

Find $N(\varepsilon/2)$ such that

$$n, m > N(\varepsilon/2) \Rightarrow |x_n - x_m| < \frac{\varepsilon}{2}$$

Let

$$\begin{aligned}\alpha_n &= \sup\{x_k : k \geq n\} \\ \beta_n &= \inf\{x_k : k \geq n\}\end{aligned}$$

Fix $m > N(\varepsilon/2)$. Then

$$\begin{aligned}k \geq m &\Rightarrow k > N(\varepsilon/2) \Rightarrow x_k < x_m + \frac{\varepsilon}{2} \\ &\Rightarrow \alpha_m = \sup\{x_k : k \geq m\} \leq x_m + \frac{\varepsilon}{2}\end{aligned}$$

Since $\alpha_m < \infty$,

$$\limsup x_n = \lim_{n\to\infty} \alpha_n \leq \alpha_m \leq x_m + \frac{\varepsilon}{2}$$

since the sequence $\{\alpha_n\}$ is decreasing. Similarly,

$$\liminf x_n \geq x_m - \frac{\varepsilon}{2}$$

Therefore,

$$0 \leq \limsup_{n\to\infty} x_n - \liminf_{n\to\infty} x_n \leq \varepsilon$$

Since ε is arbitrary,

$$\limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n \in \mathbf{R}$$

so $\lim_{n\to\infty} x_n$ exists and is real, so $\{x_n\}$ is convergent. ■

Theorem 10 (7.11) *E^n is complete for every $n \in \mathbf{N}$.*

Proof: See de la Fuente. ■

Theorem 11 (7.9) *Suppose (X, d) is a complete metric space, $Y \subseteq X$. Then $(Y, d) = (Y, d|_Y)$ is complete if and only if Y is a closed subset of X.*

Proof: Suppose (Y, d) is complete. We need to show that Y is closed. Consider a sequence $\{y_n\} \subseteq Y$ such that $y_n \rightarrow_{(X,d)} x \in X$. Then $\{y_n\}$ is Cauchy in X, hence Cauchy in Y; since Y is complete, $y_n \rightarrow_{(Y,d)} y$ for some $y \in Y$. Therefore, $y_n \rightarrow_{(X,d)} y$; by uniqueness of limits, $y = x$, so $x \in Y$, so Y is closed.

Conversely, suppose Y is closed. We need to show that Y is complete. Let $\{y_n\}$ be a Cauchy sequence in Y. Then $\{y_n\}$ is Cauchy in X, hence convergent, so $y_n \rightarrow_{(X,d)} x$ for some $x \in X$. Since Y is closed, $x \in Y$, so $y_n \rightarrow_{(Y,d)} x \in Y$, so Y is complete. ■

Theorem 12 (7.12) *Given $X \subseteq \mathbf{R}^n$, let $C(X)$ be the set of bounded continuous functions from X to $\mathbf{R}$ with*

$$\|f\|_\infty = \sup\{|f(x)| : x \in X\}$$

Then $C(X)$ is a Banach space.

Contractions

Definition 13 Let (X, d) be a nonempty complete metric space. An *operator* is a function $T : X \to X$. An operator T is a *contraction of modulus* β if $\beta < 1$ and

$$\forall_{x,y \in X}\ d(T(x), T(y)) \leq \beta d(x, y)$$

(A contraction shrinks distances by a *uniform* factor $\beta < 1$.)

Theorem 14 *Every contraction is uniformly continuous.*

Proof: Let $\delta = \frac{\varepsilon}{\beta}$. ■

A *fixed point* of an operator T is

$$x^* \in X \text{ such that } T(x^*) = x^*$$

Theorem 15 (7.16, Contraction Mapping Theorem) *Let (X, d) be a nonempty complete metric space, $T : X \to X$ a contraction with modulus $\beta < 1$. Then*

1. *T has a unique fixed point x^*.*

2. *For every $x_0 \in X$, the sequence defined by*

$$\begin{aligned} x_1 &= T(x_0) \\ x_2 &= T(x_1) \\ &\vdots \\ x_{n+1} &= T(x_n) \end{aligned}$$

   converges to x^*.

Note that the Theorem gives us an algorithm to find the fixed point of a contraction.

Proof: The proof comes in several parts:

- There can be at most one fixed point.

- The sequence $\{x_n\}$ defined in Part 2 of the statement of the theorem is Cauchy

  - We first show that the distance between the points x_n and x_{n+1} becomes very small as $n \to \infty$.

  - We then show that the distance between x_n and x_m is bounded above by a geometric series, which shows that the sequence is Cauchy.

- Since the sequence $\{x_n\}$ is Cauchy, it converges to a limit x^*.

- Because T is continuous, x^* is a fixed point.

First, we show that there is at most one fixed point. Suppose $T(x^*) = x^*$ and $T(y^*) = y^*$. Then

$$\begin{aligned} d(x^*, y^*) &= d(T(x^*), T(y^*)) \\ &\leq \beta d(x^*, y^*) \\ (1-\beta)d(x^*, y^*) &\leq 0 \\ d(x^*, y^*) &\leq 0 \end{aligned}$$

so $d(x^*, y^*) = 0$ and $x^* = y^*$.

Now, we show that the sequence $\{x_n\}$ is Cauchy, and hence converges to a limit x. Choose any $x_0 \in X$ and define x_n as described in part 2. Let $\alpha = d(x_1, x_0)$. Then

$$\begin{aligned} d(x_{n+1}, x_n) &= d(T(x_n), T(x_{n-1})) \\ &\leq \beta d(x_n, x_{n-1}) \\ &\leq \beta^2 d(x_{n-1}, x_{n-2}) \\ &\vdots \\ &\leq \beta^n d(x_1, x_0) \\ &= \beta^n \alpha \end{aligned}$$

Given $\varepsilon > 0$, by the Archimedean Property, choose $N(\varepsilon) > \frac{\log\varepsilon - \log\alpha + \log(1-\beta)}{\log\beta}$. Then since $\beta < 1$, $\log\beta < 0$ and

$$\begin{aligned} \frac{\alpha\beta^{N(\varepsilon)}}{1-\beta} &= e^{\log\left(\frac{\alpha\beta^{N(\varepsilon)}}{1-\beta}\right)} \\ &= e^{N(\varepsilon)\log\beta + \log\alpha - \log(1-\beta)} \\ &< e^{\log\varepsilon - \log\alpha + \log(1-\beta) + \log\alpha - \log(1-\beta)} \\ &= e^{\log\varepsilon} \\ &= \varepsilon \end{aligned}$$

(Note we follow the mathematics convention and denote the natural logarithm by log.) Then if $n \geq m > N(\varepsilon)$,

$$
\begin{aligned}
& d(x_n, x_m) \\
\leq\ & d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_{m+1}, x_m) \\
\leq\ & \beta^{n-1}\alpha + \beta^{n-2}\alpha + \cdots + \beta^m \alpha \\
=\ & \alpha \sum_{\ell=m}^{n-1} \beta^\ell \\
<\ & \alpha \sum_{\ell=m}^{\infty} \beta^\ell \\
=\ & \frac{\alpha\beta^m}{1-\beta} \text{ sum of a geometric series)} \\
<\ & \frac{\alpha\beta^{N(\varepsilon)}}{1-\beta} \\
<\ & \varepsilon
\end{aligned}
$$

Therefore, $\{x_n\}$ is Cauchy. Since (X, d) is complete, $x_n \to x^*$ for some $x^* \in X$.

Finally, we show that x^* is a fixed point.

$$
\begin{aligned}
T(x^*) &= T\left(\lim_{n\to\infty} x_n\right) \\
&= \lim_{n\to\infty} T(x_n) \text{ since } T \text{is continuous} \\
&= \lim_{n\to\infty} x_{n+1} \\
&= x^*
\end{aligned}
$$

so x^* is a fixed point. ■

Theorem 16 (7.18', Continuous Dependence on Parameters) *Let* (X, d) *and* (Ω, ρ) *be two metric spaces,* $T : X \times \Omega \to X$*. Let* $T_\omega : X \to X$ *be defined by*

$$T_\omega(x) = T(x, \omega)$$

Suppose (X, d) is complete, T is continuous in ω, $\beta < 1$ and

$$\forall_{\omega \in \Omega}\ T_\omega \textit{ is a contraction of modulus } \beta$$

Then the fixed point function $x^* : \Omega \to X$ defined by

$$T_\omega(x^*(\omega)) = x^*(\omega)$$

is continuous.

See the comments in the *Corrections* handout. De la Fuente's Theorem 7.18 only requires that each map T_ω be a contraction of modulus $\beta_\omega < 1$. However, his proof assumes that there is a single $\beta < 1$ such that each T_ω is a contraction of modulus β. I do not know whether de la Fuente's Theorem 7.18 is correct as stated.