Economics 204

Lecture 4–Thursday, July 30, 2009

Section 2.4, Open and Closed Sets

Definition 1 Let (X, d) be a metric space. A set $A \subseteq X$ is *open* if

$$\forall_{x \in A} \exists_{\varepsilon > 0} B_\varepsilon(x) \subseteq A$$

A set $C \subseteq X$ is *closed* if $X \setminus C$ is open.

Example: (a, b) is open in the metric space E^1 ($\mathbf{R}$ with the usual Euclidean metric). Given $x \in (a, b)$, $a < x < b$. Let

$$\varepsilon = \min\{x - a, b - x\} > 0$$

Then

$$\begin{aligned} y \in B_\varepsilon(x) &\Rightarrow y \in (x - \varepsilon, x + \varepsilon) \\ &\subseteq (x - (x - a), x + (b - x)) \\ &= (a, b) \end{aligned}$$

so $B_\varepsilon(x) \subseteq (a, b)$, so (a, b) is open.

Notice that ε depends on x; in particular, ε gets smaller as x nears the boundary of the set.

Example: In E^1, $[a, b]$ is closed. $\mathbf{R} \setminus [a, b] = (-\infty, a) \cup (b, \infty)$ is a union of two open sets, which must be open . . . .

Example: In the metric space $[0, 1]$, $[0, 1]$ is open. With $[0, 1]$ as the underlying metric space, $B_\varepsilon(0) = \{x \in [0, 1] : |x - 0| < \varepsilon = [0, \varepsilon)$. *Thus, openness and closedness depend on the underyling metric space as well as on the set.*

Example: Most sets are neither open nor closed. For example, in E^1, $[0, 1] \cup (2, 3)$ is neither open nor closed.

Example: An open set may consist of a single point. For example, if $X = \mathbf{N}$ and $d(m,n) = |m-n|$, then $B_{1/2}(1) = \{m \in \mathbf{N} : |m-1| < 1/2\} = \{1\}$.

Example: In any metric space (X,d) both $\emptyset$ and X are open, and both $\emptyset$ and X are closed. To see that $\emptyset$ is open, note that the statement

$$\forall_{x\in\emptyset}\exists_{\varepsilon>0}\ B_\varepsilon(x) \subseteq \emptyset$$

is vacuously true since there aren't any $x \in \emptyset$. To see that X is open, note that since $B_\varepsilon(x)$ is by definition $\{z \in X : d(z,x) < \varepsilon\}$, it is trivially contained in X. Since $\emptyset$ is open, X is closed; since X is open, $\emptyset$ is closed.

Example: Open balls are open sets. Suppose $y \in B_\varepsilon(x)$. Then $d(x,y) < \varepsilon$. Let $\delta = \varepsilon - d(x,y) > 0$. If $d(z,y) < \delta$, then

$$\begin{aligned} d(z,x) &\leq d(z,y) + d(y,x) \\ &< \delta + d(x,y) \\ &= \varepsilon - d(x,y) + d(x,y) \\ &= \varepsilon \end{aligned}$$

so $B_\delta(y) \subseteq B_\epsilon(x)$, so $B_\varepsilon(x)$ is open.

Theorem 2 (4.2) *Let (X,d) be a metric space. Then*

1. *$\emptyset$ and X are both open, and both closed.*
2. *The union of an arbitrary (finite, countable, or uncountable) collection of open sets is open.*
3. *The intersection of a finite collection of open sets is open.*

Proof:

1. We have already done this.

2. Suppose $\{A_\lambda\}_{\lambda\in\Lambda}$ is a collection of open sets.

$$\begin{aligned} x \in \bigcup_{\lambda\in\Lambda} A_\lambda &\Rightarrow \exists_{\lambda_0\in\Lambda}\ x \in A_{\lambda_0} \\ &\Rightarrow \exists_{\varepsilon>0}\ B_\varepsilon(x) \subseteq A_{\lambda_0} \subseteq \bigcup_{\lambda\in\Lambda} A_\lambda \end{aligned}$$

so $\cup_{\lambda\in\Lambda} A_\lambda$ is open.

3. Suppose $A_1, \ldots, A_n \subseteq X$ are open sets. If $x \in \cap_{i=1}^n A_i$, then

$$x \in A_1, x \in A_2, \ldots, x \in A_n$$

so

$$\exists_{\varepsilon_1>0,\ldots,\varepsilon_n>0}\ B_{\varepsilon_1}(x) \subseteq A_1, \ldots, B_{\varepsilon_n}(x) \subseteq A_n$$

Let

$$\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_n\} > 0$$

(*Aside: this is where we need the fact that we are taking a finite union. The infimum of an infinite set of positive numbers could be zero.*)

Then

$$B_\varepsilon(x) \subseteq B_{\varepsilon_1}(x) \subseteq A_1, \ldots, B_\varepsilon(x) \subseteq B_{\varepsilon_n}(x) \subseteq A_n$$

so

$$B_\varepsilon(x) \subseteq \bigcap_{i=1}^n A_i$$

which proves that $\cap_{i=1}^n A_i$ is open.

■

Definition 3
- int A: the *interior* of A, the largest open set contained in A (the union of all open sets contained in A)
- $\bar{A}$: the *closure* of A, the smallest closed set containing A (the intersection of all closed sets containing A)

- $\operatorname{ext} A$: the *exterior* of A, the largest open set contained in $X \setminus A$.

- ∂A: the *boundary* of A, $\overline{(X \setminus A)} \cap \bar{A}$

Theorem 4 (4.13) *A set A in a metric space (X, d) is closed if and only if*

$$\{x_n\} \subset A, \{x_n\} \to x \in X \Rightarrow x \in A$$

Proof: (*This is different from the proof in de la Fuente: he puts the meat of the proof into Theorem 4.12*) Suppose A is closed. Then $X \setminus A$ is open. Consider a convergent sequence $\{x_n\} \to x \in X$, with $x_n \in A$ for all n. If $x \notin A$, $x \in X \setminus A$, so there is some $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq X \setminus A$. Since $x_n \to x$, there exists $N(\varepsilon)$ such that

$$\begin{aligned} n > N(\varepsilon) \quad &\Rightarrow \quad x_n \in B_\varepsilon(x) \\ &\Rightarrow \quad x_n \in X \setminus A \\ &\Rightarrow \quad x_n \notin A \end{aligned}$$

contradiction. Therefore,

$$\{x_n\} \subset A, \{x_n\} \to x \in X \Rightarrow x \in A$$

Conversely, suppose

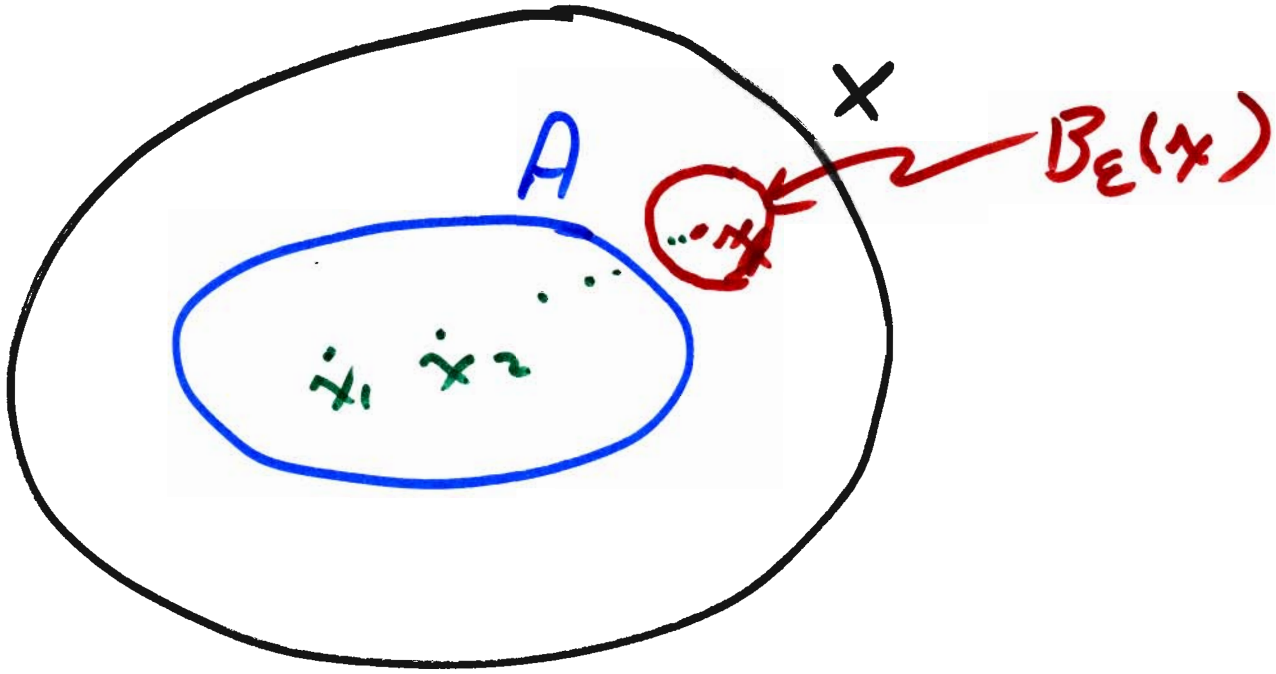
$$\{x_n\} \subset A, \{x_n\} \to x \in X \Rightarrow x \in A$$

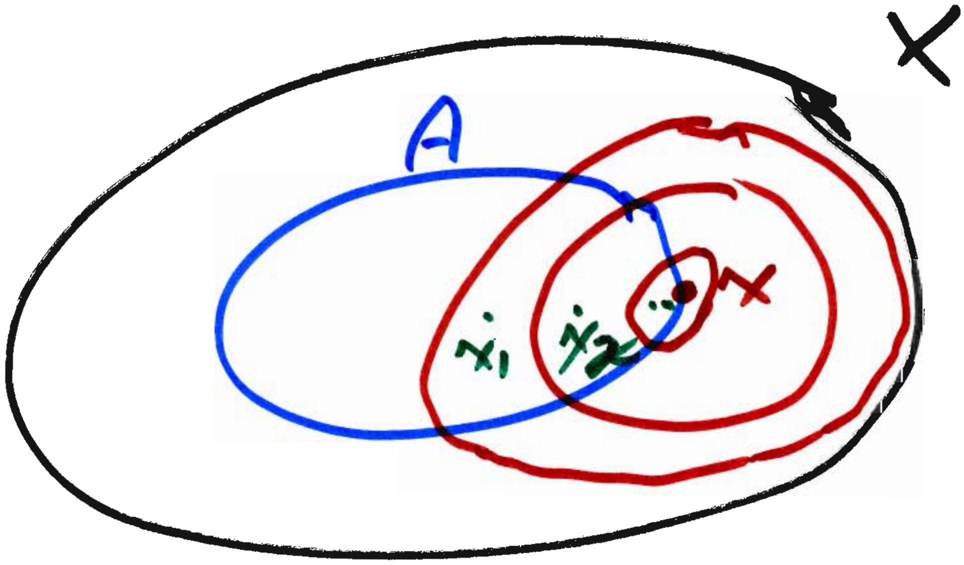
We need to show that A is closed, i.e. $X \setminus A$ is open. Suppose not, so $X \setminus A$ is not open. Then there exists $x \in X \setminus A$ such that for every $\varepsilon > 0$,

$$B_\varepsilon(x) \not\subseteq X \setminus A$$

so there exists $y \in B_\varepsilon(x)$ such that $y \notin X \setminus A$, so $y \in A$ so

$$B_\varepsilon(x) \bigcap A \neq \emptyset$$


X
A
$B_\varepsilon(x)$
x
x_1 x_2


X
A
x
x_1 x_2

Construct a sequence $\{x_n\}$ as follows: for each n, choose $x_n \in B_{\frac{1}{n}}(x) \cap A$. Given $\varepsilon > 0$, we can find $N(\varepsilon)$ such that $N(\varepsilon) > \frac{1}{\varepsilon}$ by the Archimedean Property, so $n > N(\varepsilon) \Rightarrow \frac{1}{n} < \frac{1}{N(\varepsilon)} < \varepsilon$, so $x_n \to x$. Then $\{x_n\} \subseteq A$, $\{x_n\} \to x$, so $x \in A$, contradiction. Therefore, $X \setminus A$ is open, so A is closed. ■

Section 2.5: Limits of Functions *Read this on your own.* Note that we may have $\lim_{x\to a} f(x) = y$ even though

- f is not defined at a; or
- f is defined at a but $f(a) \neq y$.

The existence and value of the limit depends on values of f *near* a but not *at* a.

Section 2.6: Continuity in Metric Spaces

Definition 5 Let (X, d) and (Y, ρ) be metric spaces, $f : X \to Y$. f is *continuous at a point* $x_0 \in X$ if

$$\forall_{\varepsilon>0}\exists_{\delta(x_0,\varepsilon)>0}\ d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

f is *continuous* if it is continuous at every element of its domain.

Note: δ depends on x_0, ε. This is a straightforward generalization of the definition of continuity in $\mathbf{R}$. Continuity at x_0 requires:

- $f(x_0)$ is defined; and
- either
  - x_0 is an isolated point of X, i.e. $\exists_{\varepsilon>0} B_\varepsilon(x) = \{x\}$; or
  - $\lim_{x\to x_0} f(x)$ exists and equals $f(x_0)$

(*We will go out of order.*)

Define $f^{-1}(A) = \{x \in X : f(x) \in A\}$

Theorem 6 (6.14) *Let (X, d) and (Y, ρ) be metric spaces, $f : X \to Y$. Then f is continuous if and only if*

$$\forall_{A \subseteq Y}\ A \textit{ open in } Y \Rightarrow f^{-1}(A) \textit{ is open in } X$$

Proof: (*We give a direct proof; de la Fuente works via closed sets*)

Suppose f is continuous. Given $A \subseteq Y$, A open, we must show that $f^{-1}(A)$ is open in X. Suppose $x_0 \in f^{-1}(A)$. Let $y_0 = f(x_0) \in A$. Since A is open, we can find $\varepsilon > 0$ such that $B_\varepsilon(y_0) \subseteq A$. Since f is continuous, there exists $\delta > 0$ such that

$$\begin{aligned} d(x, x_0) < \delta &\Rightarrow \rho(f(x), f(x_0)) < \varepsilon \\ &\Rightarrow f(x) \in B_\varepsilon(y_0) \\ &\Rightarrow f(x) \in A \\ &\Rightarrow x \in f^{-1}(A) \end{aligned}$$

so $B_\delta(x_0) \subseteq f^{-1}(A)$, so $f^{-1}(A)$ is open.

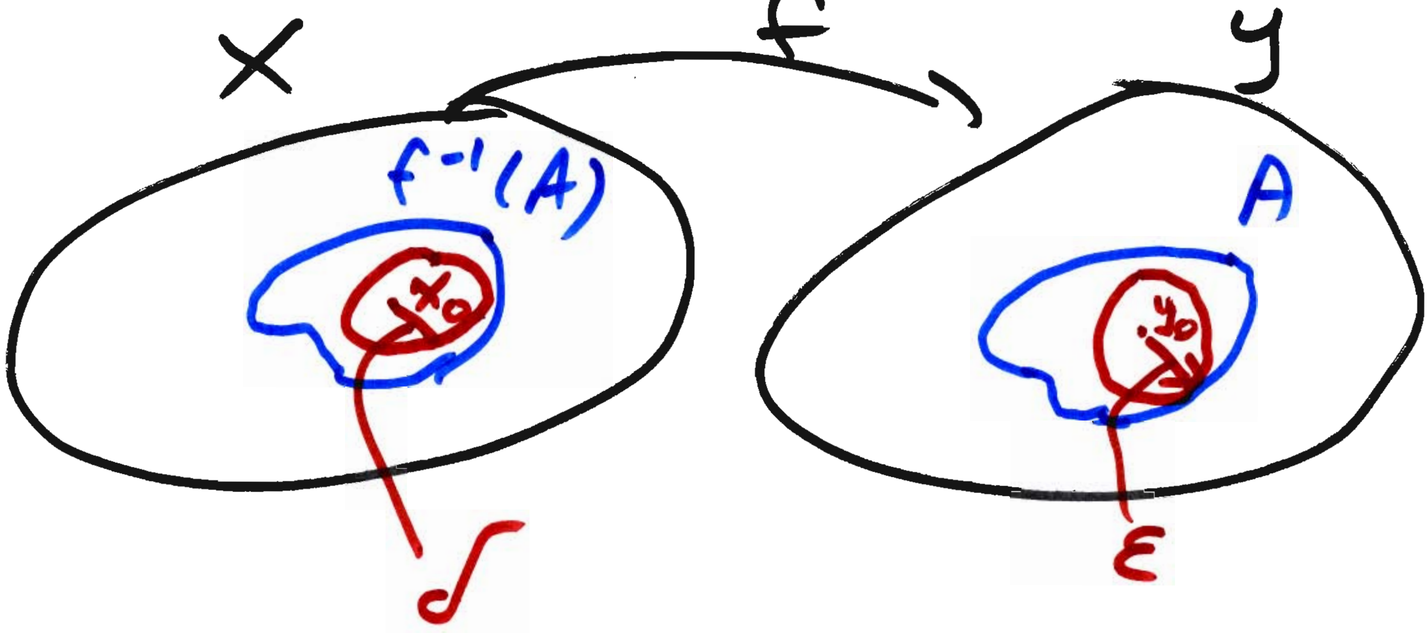
Conversely, suppose

$$\forall_{A \subseteq Y}\ A \text{ open in } Y \Rightarrow f^{-1}(A) \text{ is open in } X$$

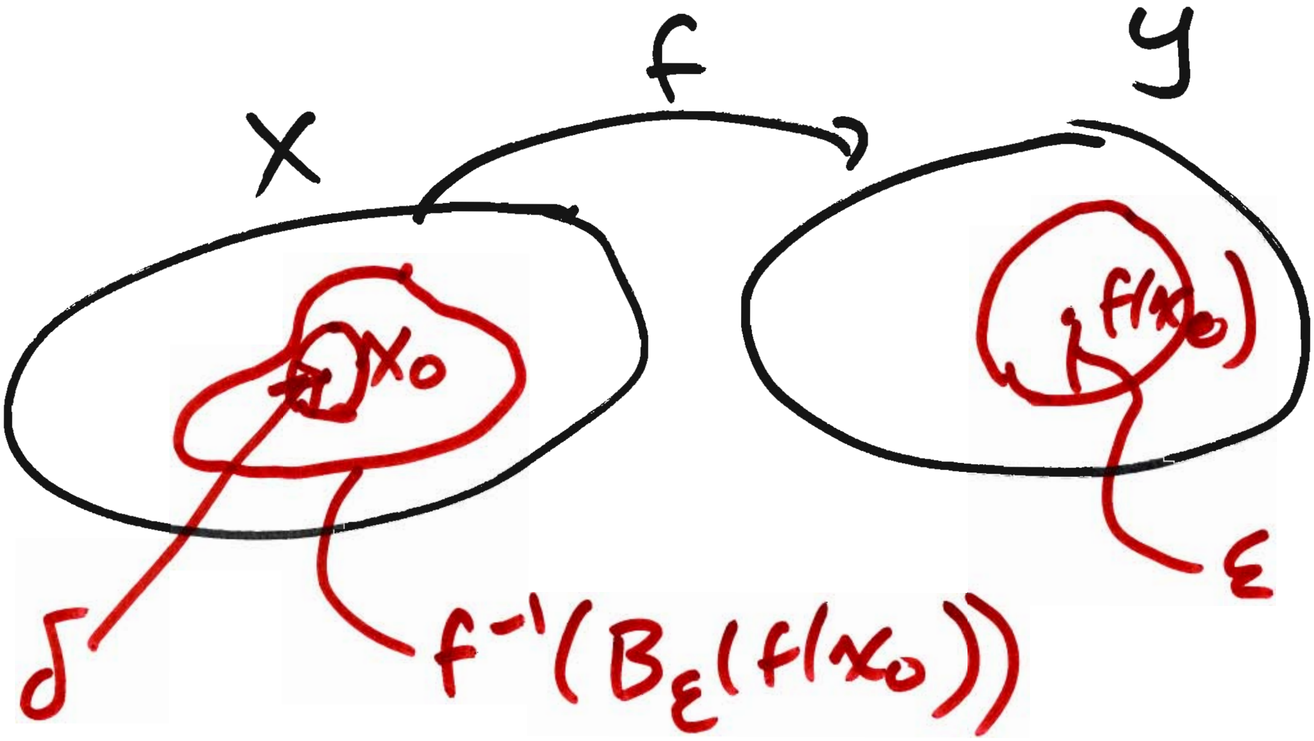
We need to show that f is continuous. Let $x_0 \in X$, $\varepsilon > 0$. Let $A = B_\varepsilon(f(x_0))$. A is an open ball, hence an open set, so $f^{-1}(A)$ is open in X. $x_0 \in f^{-1}(A)$, so there exists $\delta > 0$ such that $B_\delta(x_0) \subseteq f^{-1}(A)$.

$$\begin{aligned} d(x, x_0) < \delta &\Rightarrow x \in B_\delta(x_0) \\ &\Rightarrow x \in f^{-1}(A) \\ &\Rightarrow f(x) \in A \\ &\Rightarrow \rho(f(x), f(x_0)) < \varepsilon \end{aligned}$$

Thus, we have shown that f is continuous at x_0; since x_0 is an arbitrary point in X, f is continuous.■



X
f
Y
$f^{-1}(A)$
A
x_0
y_0
δ
ε



X
f
Y
x_0
$f(x_0)$
δ
ε
$f^{-1}(B_\varepsilon(f(x_0)))$

Theorem 7 (Slightly weaker version of 6.10) *Let (X, d_X), (Y, d_Y) and (Z, d_Z) be metric spaces. If $f : X \to Y$ and $g : Y \to Z$ are continuous, then $g \circ f : X \to Z$ is continuous.*

Proof: Suppose $A \subseteq Z$ is open. Since g is continuous, $g^{-1}(A)$ is open in Y; since f is continuous, $f^{-1}(g^{-1}(A))$ is open in X.

We claim that

$$f^{-1}(g^{-1}(A)) = (g \circ f)^{-1}(A)$$

Observe

$$\begin{aligned} x \in f^{-1}(g^{-1}(A)) &\Leftrightarrow f(x) \in g^{-1}(A) \\ &\Leftrightarrow g(f(x)) \in A \\ &\Leftrightarrow (g \circ f)(x) \in A \\ &\Leftrightarrow x \in (g \circ f)^{-1}(A) \end{aligned}$$

which establishes the claim. This shows that $(g \circ f)^{-1}(A)$ is open in X, so $g \circ f$ is continuous. ■

Definition 8 [Uniform Continuity] (*Important*) Suppose $f : (X, d) \to (Y, \rho)$. f is continuous means

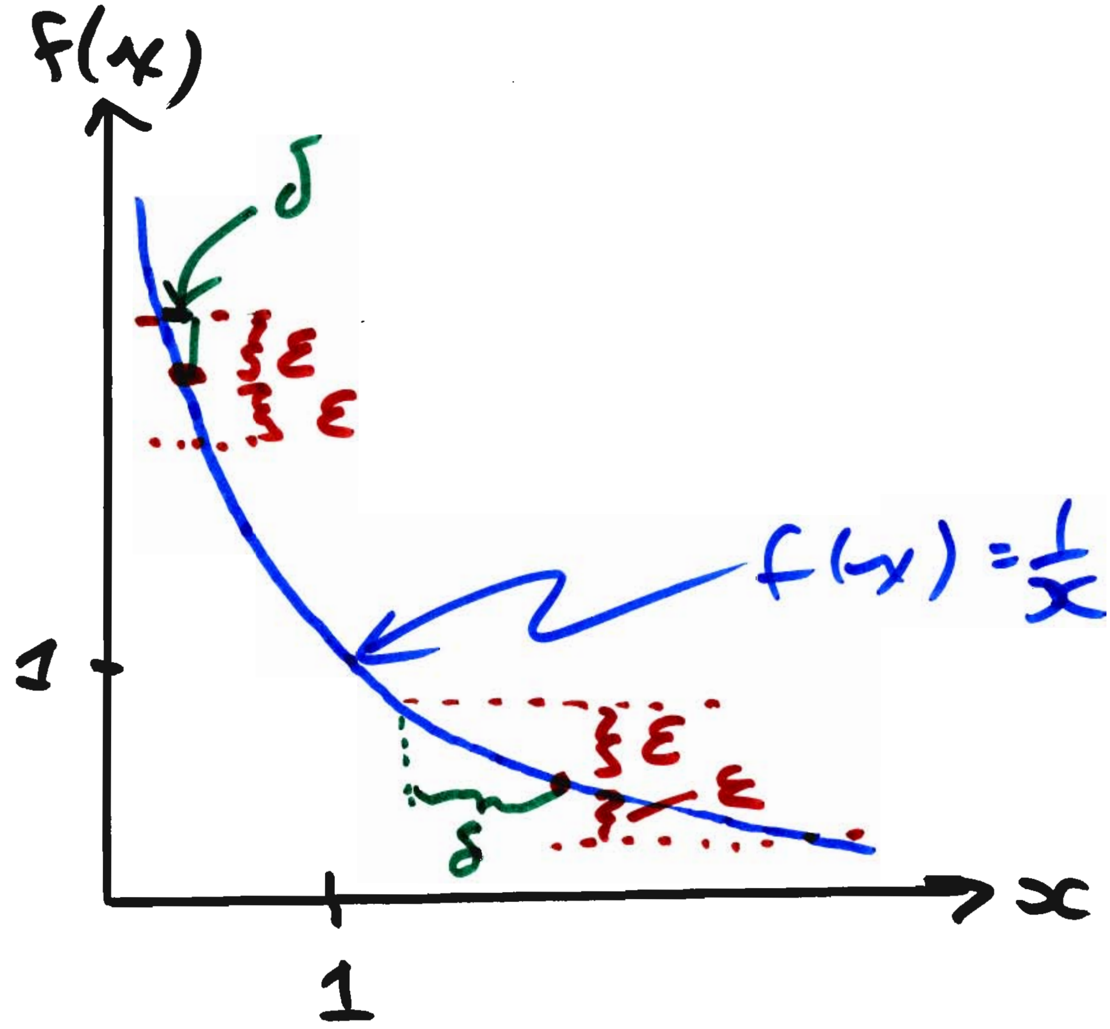
$$\forall_{x_0 \in X, \varepsilon > 0} \exists_{\delta(x_0, \varepsilon) > 0} \ d(x, x_0) < \delta(x_0, \varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

f is uniformly continuous means

$$\forall_{\varepsilon > 0} \exists_{\delta(\varepsilon) > 0} \forall_{x_0 \in X} \ d(x, x_0) < \delta(\varepsilon) \Rightarrow \rho(f(x), f(x_0)) < \varepsilon$$

Example: Consider

$$f(x) = \frac{1}{x}, \ \ x \in (0, 1]$$



f(x)
δ
ε
ε
f(x) = 1/x
1
ε
ε
δ
x
1

f is continuous from Math 1A. We will show that f is not uniformly continuous. Fix $\varepsilon > 0$ and $x_0 \in (0, 1]$.

If $x = \frac{x_0}{1+\varepsilon x_0}$, then

$$\begin{aligned} 1 + \varepsilon x_0 &> 1 \\ x = \frac{x_0}{1 + \varepsilon x_0} &< x_0 \\ \frac{1}{x} - \frac{1}{x_0} &> 0 \\ |f(x) - f(x_0)| &= \left|\frac{1}{x} - \frac{1}{x_0}\right| \\ &= \frac{1}{x} - \frac{1}{x_0} \\ &= \frac{1 + \varepsilon x_0}{x_0} - \frac{1}{x_0} \\ &= \frac{\varepsilon x_0}{x_0} \\ &= \varepsilon \end{aligned}$$

Thus, $\delta(x_0, \varepsilon)$ must be chosen small enough so that

$$\left|\frac{x_0}{1 + \varepsilon x_0} - x_0\right| \geq \delta(x_0, \varepsilon)$$

$$\begin{aligned} \delta(x_0, \varepsilon) &\leq x_0 - \frac{x_0}{1 + \varepsilon x_0} \\ &= \frac{\varepsilon (x_0)^2}{1 + \varepsilon x_0} \\ &< \varepsilon (x_0)^2 \end{aligned}$$

which converges to zero as $x_0 \to 0$, so there is no $\delta(\varepsilon)$ which will work for all $x_0 \in (0, 1]$.

Example: If $f'(x)$ is defined and uniformly bounded on an interval $[a, b]$, then $f(x)$ is uniformly continuous on $[a, b]$. However, even a function with an unbounded derivative may be uniformly continuous. Consider

$$f(x) = \sqrt{x}, \ \ x \in [0,1]$$

f is continuous from Math 1A. We will show that f is uniformly continuous. Given $\varepsilon > 0$, let $\delta = \varepsilon^2$. Then given any $x_0 \in [0,1]$, $|x - x_0| < \delta$ implies by the Fundamental Theorem of Calculus

$$\begin{aligned}
|f(x) - f(x_0)| &= \left| \int_{x_0}^{x} \frac{1}{2\sqrt{t}}\, dt \right| \\
&\leq \int_0^{|x-x_0|} \frac{1}{2\sqrt{t}}\, dt \\
&= \sqrt{|x - x_0|} \\
&< \sqrt{\delta} \\
&= \sqrt{\varepsilon^2} \\
&= \varepsilon
\end{aligned}$$

Thus, $f(x)$ is uniformly continuous on $[0,1]$, even though $f'(x) \to \infty$ as $x \to 0$.

Definition 9 [Lipschitz Functions] Let X, Y be normed vector space, $E \subseteq X$. $f : X \to Y$ is *Lipschitz on* E if

$$\exists_{K>0} \forall_{x,z \in E} \ \|f(x) - f(z)\|_Y \leq K \|x - z\|_X$$

f is *locally Lipschitz* on E if

$$\forall_{x_0 \in E} \exists_{\varepsilon > 0} \ f \text{ is Lipschitz on } B_\varepsilon(x_0) \cap E$$

$$\begin{aligned}
C^1 \Rightarrow \text{locally Lipschitz} &\Rightarrow \text{continuous} \\
\text{Lipschitz} &\Rightarrow \text{uniformly continuous}
\end{aligned}$$

Homeomorphisms:

Definition 10 Let (X, d) and (Y, ρ) be metric spaces. A function $f : X \to Y$ is called a *homeomorphism* if it is one-to-one and continuous, and its inverse function is continuous on $f(X)$.

(*Aside: this is not the standard definition; most texts also require that the function be onto. See the Corrections handout for a correction to Theorem 6.21*)

Now suppose that f is a homeomorphism and $U \subset X$.

$$
\begin{aligned}
y \in \left(f^{-1}\right)^{-1}(U) &\Leftrightarrow f^{-1}(y) \in U \\
&\Leftrightarrow y \in f(U) \\
U \text{ open in } X &\Rightarrow \left(f^{-1}\right)^{-1}(U) \text{ is open in } (f(X), \rho) \\
&\Rightarrow f(U) \text{ is open in } (f(X), \rho)
\end{aligned}
$$

This says that X and $\left(f(X), \rho|_{f(X)}\right)$ are identical in terms of properties that can be characterized solely in terms of open sets; such properties are called "topological properties."