**Economics 204 Summer/Fall 2008**

**Lecture 1–Monday July 27, 2009**

**Bob Anderson**

**Vladimir Asriyan**

**Hui Zheng**

**Website:** http://emlab.berkeley.edu/users/anderson/Econ204

/204index.html

**Lectures** will often run past 3:00, don't schedule things before 3:30.

**Sections:** 9-10:30, 10:30-12:00 in 608-7 Evans; please split up evenly.

**Prerequisites:** Berkeley Math 53-54 or equivalent:

- four semesters of college mathematics
- linear algebra
- multivariable calculus
- rigorous: theorems stated carefully and some proofs given
- stream for engineers and scientists

**Grading in First Year Econ Grad Classes:**

- median grade B+: solid command of material
- A and A- are really good grades; A+ truly exceptional
- B: ready to go on to further work. B in 204, ready to do 201A-B, 202A-B,240A-B
- B-: really marginal, but we're not going to make you take it again. B- in 204, you're going to have a very hard time in 201A-B. Should seriously consider taking Math 53-54 this year,

retake 204 next year, delay 201A-B for a year. B- is a passing grade for Berkeley grad students, but grad students are required to maintain a B average.

- C: definitely not ready for 201A-B, 202A-B, 240A-B. Take Math 53-54 this year, retake 204 next year, defer 201A-B. 204 with at least a B- will be an *enforced* prerequisite for 201A-B.
- F: didn't take final exam. F's do horrible things to your GPA, so make sure you *withdraw* if you don't take the final.

**Goals:**

- Reduce heterogeneity of math backgrounds of students in Econ grad classes
- Advance everyone's math skills
- Challenge everyone; consequently, not everyone will understand everything
- Develop math skills needed to work as a professional economist (text is from Penn's math class for graduate students)
- Ability to read purported proofs and determine whether or not they are correct. Essential to reading the models underlying mainstream theoretical, empirical, and experimental economics papers.
- Ability to compose simple proofs; essential to writing models in mainstream theoretical, empirical, and experimental economics papers.
- Cover material in real analysis and linear algebra at a moderate level of abstraction (considerably more abstract than Math 53-54).
- **Not** to review Math 53-54. If your background in this material is weak, you should take Math 53-54 and come back in a year.

**Learning by Doing:**

- You don't learn Math just by listening in lecture or reading the book.
- You *do* learn Math by doing problems.
- Working in groups is strongly encouraged.
- Try all the problems yourself before meeting with study group.
- You must write up your own solution. If you have to copy someone else's, it means you don't understand it (and will not be able to answer related questions on the exam).
- You don't really understand something until you can explain it to someone else.

**Section 1.2, Methods of Proof:**

**What is a proof?** formal definition in mathematical logic, a formal proof is long and unreadable. In practice, recognize a proof when you see one.

**Methods of Proof:**

- deduction
- contraposition
- induction
- contradiction

**Proof by Deduction:** A list of statements, the last of which is the statement to be proven. Each statement in the list is either

- an axiom: a fundamental assumption about mathematics, or part of definition of the object under study; or
- a previously established theorem; or
- follows from previous statements in the list by a valid rule of inference

**Example:** Prove that the function $f(x) = x^2$ is continuous at $x = 5$.

Recall from one-variable calculus that $f(x) = x^2$ is continuous at $x = 5$ means

$$\forall_{\varepsilon>0}\exists_{\delta>0}|x-5| < \delta \Rightarrow |f(x) - f(5)| < \varepsilon$$

"For every $\varepsilon > 0$ there exists a $\delta > 0$ such that whenever $x$ is within $\delta$ of 5, $f(x)$ is within $\varepsilon$ of $f(5)$."

The proof must systematically verify that this definition is satisfied.

**Proof:** Suppose we're given $\varepsilon > 0$. Let

$$\delta = \min\left\{1, \frac{\varepsilon}{11}\right\} > 0$$

*Where did that come from ?...*

Suppose $|x-5| < \delta$. Since $\delta \leq 1$, $4 < x < 6$, so $9 < x + 5 < 11$, so $|x+5| < 11$. Then

$$\begin{aligned}
|f(x) - f(5)| &= |x^2 - 25| \\
&= |(x-5)(x+5)| \\
&= |x-5||x+5| \\
&< \delta \cdot 11 \\
&\leq \frac{\varepsilon}{11} \cdot 11 \\
&= \varepsilon
\end{aligned}$$

Thus, we have shown that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $|x-5| < \delta \Rightarrow |f(x) - f(5)| < \varepsilon$, so $f(x) = x^2$ is continuous at $x = 5$. $\blacksquare$

**Proof by Contraposition:**

$\neg P$ means "P is false."

$P \wedge Q$ means "$P$ is true *and* $Q$ is true."

$P \vee Q$ means "$P$ is true *or* $Q$ is true (or possibly both)."

$\neg P \wedge Q$ means $(\neg P) \wedge Q$; $\neg P \vee Q$ means $(\neg P) \vee Q$.

$P \Rightarrow Q$ means "whenever $P$ is satisfied, $Q$ is also satisfied." Formally, $P \Rightarrow Q$ is equivalent to $\neg P \vee Q$.

The *contrapositive* of the statement $P \Rightarrow Q$ is the statement

$$\neg Q \Rightarrow \neg P$$

**Theorem 1** *$P \Rightarrow Q$ is true if and only if $\neg Q \Rightarrow \neg P$ is true.*

**Proof:** Suppose $P \Rightarrow Q$ is true. Then either $P$ is false, or $Q$ is true (or possibly both). Therefore, either $\neg P$ is true, or $\neg Q$ is false (or possibly both), so $\neg(\neg Q) \vee (\neg P)$ is true, $\neg Q \Rightarrow \neg P$ is true.

Conversely, suppose $\neg Q \Rightarrow \neg P$ is true. Then either $\neg Q$ is false, or $\neg P$ is true (or possibly both), so either $Q$ is true, or $P$ is false (or possibly both), so $\neg P \vee Q$ is true, so $P \Rightarrow Q$ is true. $\blacksquare$

See the book for an example of the use of proof by contraposition.

**Proof by Induction:**

An example:

**Theorem 2** *For every $n \in \mathbf{N}_0 = \{0, 1, 2, 3, \ldots\}$,*

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

*i.e.* $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$.

**Proof:**

*Base step* $n = 0$: L.S. $= \sum_{k=1}^{0} k =$ the empty sum $= 0$. R.S. $= \frac{0 \cdot 1}{2} = 0$ so the theorem is true for $n = 0$.

*Induction step:* Suppose

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \text{ for some } n$$

We must show that

$$\sum_{k=1}^{n+1} k = \frac{(n+1)((n+1)+1)}{2}$$

$$\begin{aligned}
\text{L.S.} &= \sum_{k=1}^{n+1} k \\
&= \sum_{k=1}^{n} k + (n+1) \\
&= \frac{n(n+1)}{2} + (n+1) \text{ by the Induction hypothesis} \\
&= (n+1)\left(\frac{n}{2} + 1\right) \\
&= \frac{(n+1)(n+2)}{2} \\
\text{R.S.} &= \frac{(n+1)((n+1)+1)}{2} \\
&= \frac{(n+1)(n+2)}{2} \\
&= \text{L.S.}
\end{aligned}$$

so by mathematical induction, $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ for all $n \in \mathbf{N}_0$. ■

**Proof by Contradiction:**

**Theorem 3** *There is no rational number $q$ such that $q^2 = 2$.*

**Proof:** Suppose $q^2 = 2$, $q \in \mathbf{Q}$. We can write $q = \frac{m}{n}$ for some $m, n \in \mathbf{Z}$. Moreover, we can assume that $m$ and $n$ have no common factor; if they did, we could divide it out. *(Aside: this is actually a subtle point. We are using the fact that the expression of a natural number as a product of primes is unique.)*

$$2 = q^2 = \frac{m^2}{n^2}$$

Therefore, $m^2 = 2n^2$, so $m^2$ is even.

We claim that $m$ is even. If not (*Aside: This is a proof by contradiction within a proof by contradiction!*) $m$ is odd, so $m = 2p + 1$ for some $p \in \mathbf{Z}$. Then

$$\begin{aligned} m^2 &= (2p+1)^2 \\ &= 4p^2 + 4p + 1 \\ &= 2(2p^2 + 2p) + 1 \end{aligned}$$

which is odd, contradiction. Therefore, $m$ is even, so $m = 2r$ for some $r \in \mathbf{Z}$.

$$\begin{aligned} 4r^2 &= (2r)^2 \\ &= m^2 \\ &= 2n^2 \\ n^2 &= 2r^2 \end{aligned}$$

so $n^2$ is even, which implies (by the argument given above) that $n$ is even. Therefore, $n = 2s$ for some $s \in \mathbf{Z}$, so $m$ and $n$ have a common factor, namely 2, contradiction. Therefore, there is no rational number $q$ such that $q^2 = 2$. ■

**Section 1.3, Equivalence Relations**

**Definition 4** A *binary relation* $R$ on a set $X$ is a subset of $X \times X$. Write $xRy$ as an abbreviation for $(x, y) \in R$.

Example: Indifference: $x \sim y$.

**Definition 5** $R$ is

- *reflexive* if $\forall_{x \in X} xRx$
- *symmetric* if $\forall_{x,y \in X} xRy \Leftrightarrow yRx$
- *transitive* if $\forall_{x,y,z \in X} (xRy \wedge yRz) \Rightarrow xRz$
- an *equivalence relation* if it is reflexive, symmetric and transitive

**Definition 6** Given an equivalence relation $R$, write

$$[x] = \{y \in X : xRy\}$$

$[x]$ is called the *equivalence class containing* $x$.

The following theorem says that the equivalence classes form a *partition* of $X$; every element of $X$ belongs to exactly one equivalence class.

**Theorem 7** *Let $R$ be an equivalence relation on $X$. Then $\forall_{x \in X}\ x \in [x]$. Given $x, y \in X$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.*

**Proof:** If $x \in X$, then $xRx$ because $R$ is reflexive, so $x \in [x]$. Suppose $x, y \in X$. If $[x] \cap [y] = \emptyset$, we're done.

So suppose $[x] \cap [y] \neq \emptyset$. We must show that $[x] = [y]$, i.e. that the elements of $[x]$ are exactly the same as the elements of $[y]$. Choose $z \in [x] \cap [y]$.

$z \in [x]$, so $xRz$, so $zRx$ since $R$ is symmetric.

$z \in [y]$, so $yRz$, so $zRy$ since $R$ is symmetric.

Suppose $w \in [x]$. Then $xRw$ and $zRx$, so $zRw$ (transitivity), so $wRz$ (symmetry) and $zRy$, so $wRy$ (transitivity), so $yRw$ (symmetry), so $w \in [y]$, which shows that $[x] \subseteq [y]$.

Similarly, $[y] \subseteq [x]$, so $[x] = [y]$.$\blacksquare$

The set of equivalence classes is called the *quotient* of $X$ with respect to $R$.

**Section 1.4, Cardinality**

Two sets $A, B$ are *numerically equivalent* (*have the same cardinality*) if there is a bijection $f : A \to B$, i.e. $f$ is a function, $f$ is 1-1 ($a \neq a' \Rightarrow f(a) \neq f(a')$), and $f$ is onto ($\forall_{b \in B} \exists_{a \in A} f(a) = b$). Roughly speaking, elements of the sets can be paired off.

**Finite Sets:** Numerically equivalent to $\{1, \ldots, n\}$ for some $n$.

$\{1, 2, \ldots, 25\}$ is numerically equivalent to $\{2, 4, 6, \ldots, 50\}$ under the function $f(n) = 2n$.

**Infinite Sets:** Sets which are not finite.

Sets that are numerically equivalent to $\mathbf{N}$ are called *countable*.

Example: $\mathbf{Z}$ is countable.

$$Z = \{0, 1, -1, 2, -2, \ldots\}$$

Define $f : \mathbf{N} \to \mathbf{Z}$ by

$$\begin{aligned} f(1) &= 0 \\ f(2) &= 1 \\ f(3) &= -1 \\ &\vdots \\ f(n) &= (-1)^n \left\lfloor \frac{n}{2} \right\rfloor \end{aligned}$$

where $\lfloor x \rfloor$ is the greatest integer less than or equal to $x$.

$f$ is one-to-one and onto.

Notice $\mathbf{Z} \supset \mathbf{N}$, $\mathbf{Z} \neq \mathbf{N}$; indeed, $\mathbf{Z} \setminus \mathbf{N}$ is infinite!

"One half of the elements of $\mathbf{Z}$ are in $\mathbf{N}$" is not a meaningful statement.

**Theorem 8** $\mathbf{Q}$ *is countable.*

**"Picture Proof":**

$$\begin{aligned} \mathbf{Q} &= \left\{ \frac{m}{n} : m, n \in \mathbf{Z}, n \neq 0 \right\} \\ &= \left\{ \frac{m}{n} : m \in \mathbf{Z}, n \in \mathbf{N} \right\} \end{aligned}$$

$$\begin{array}{cc|ccccccccc}
 & & & & & & m & & & & \\
 & & 0 & & 1 & & -1 & & 2 & & -2 \\
\hline
 & 1 & 0 & \rightarrow & 1 & & -1 & \rightarrow & 2 & & -2 \\
 & & & \swarrow & & \nearrow & & \swarrow & & \nearrow & \\
 & 2 & 0 & & \frac{1}{2} & & -\frac{1}{2} & & 1 & & -1 \\
 & & \downarrow & \nearrow & & \swarrow & & \nearrow & & & \\
n & 3 & 0 & & \frac{1}{3} & & -\frac{1}{3} & & \frac{2}{3} & & -\frac{2}{3} \\
 & & & \swarrow & & \nearrow & & & & & \\
 & 4 & 0 & & \frac{1}{4} & & -\frac{1}{4} & & \frac{1}{2} & & -\frac{1}{2} \\
 & & \downarrow & \nearrow & & & & & & & \\
 & 5 & 0 & & \frac{1}{5} & & -\frac{1}{5} & & \frac{2}{5} & & -\frac{2}{5}
\end{array}$$

Go back and forth on upward-sloping diagonals, omitting the repeats:

$$\begin{aligned} f(1) &= 0 \\ f(2) &= 1 \\ f(3) &= \frac{1}{2} \\ f(4) &= -1 \\ &\vdots \end{aligned}$$

$f : \mathbf{N} \to \mathbf{Q}$, $f$ one-to-one and onto.

*Although* $\mathbf{Q}$ *appears to be much larger than* $\mathbf{N}$, *in fact they are the same size!*