Economics 201B

Nonconvex Preferences and Approximate Equilibria

1 The Shapley-Folkman Theorem

The Shapley-Folkman Theorem is an elementary result in linear algebra, but it is apparently unknown outside the mathematical economics literature. It is closely related to Caratheodory's Theorem, a linear algebra result which is well known to mathematicians. The Shapley-Folkman Theorem was first published in Starr [3], an important early paper on existence of approximate equilibria with nonconvex preferences.

Theorem 1.1 (Caratheodory) *Suppose* $x \in \text{con}\ A$, *where* $A \subset \mathbf{R}^L$. *Then there are points* $a_1, \ldots, a_{L+1} \in A$ *such that* $x \in \text{con}\ \{a_1, \ldots, a_{L+1}\}$.

Theorem 1.2 (Shapley-Folkman) *Suppose* $x \in \text{con}\ (A_1 + \cdots + A_I)$, *where* $A_i \subset \mathbf{R}^L$. *Then we may write* $x = a_1 + \cdots + a_I$, *where* $a_i \in \text{con}\ A_i$ *for all* i *and* $a_i \in A_i$ *for all but* L *values of* i.

We derive both Caratheodory's Theorem and the Shapley-Folkman Theorem from the following lemma:

Lemma 1.3 *Suppose* $x \in \text{con}\ (A_1 + \cdots + A_I)$ *where* $A_i \subset \mathbf{R}^L$. *Then we may write*

$$x = \sum_{i=1}^{I} \sum_{j=0}^{m_i} \lambda_{ij} a_{ij} \tag{1}$$

with $\sum_{i=1}^{I} m_i \leq L$; $a_{ij} \in A_i$ *and* $\lambda_{ij} > 0$ *for each* i, j; *and* $\sum_{j=0}^{m_i} \lambda_{ij} = 1$ *for each* i.

Proof:

1. Suppose $x \in \text{con}\,(A_1 + \cdots + A_I)$. Then we may write

$$x = \sum_{j=0}^{m} \lambda_j \sum_{i=1}^{I} a_{ij} = \sum_{i=1}^{I} \sum_{j=0}^{m} \lambda_j a_{ij} \tag{2}$$

with $\lambda_j > 0$, $\sum_{j=0}^{m} \lambda_j = 1$. Letting $\lambda_{ij} = \lambda_j$ and $m_i = m$ for each i, we have an expression for x in the form of equation 1.

2. Suppose we have any expression for x in the form of equation 1 with $\sum_{i=1}^{I} m_i > L$. Then the set

$$\{a_{ij} - a_{i0} : 1 \leq i \leq I,\ 1 \leq j \leq m_i\} \tag{3}$$

contains $\sum_{i=1}^{I} m_i > L$ vectors in $\mathbf{R}^L$, and hence is linearly dependent. Therefore, we can find β_{ij} not all zero such that

$$\sum_{i=1}^{I} \sum_{j=1}^{m_i} \beta_{ij}(a_{ij} - a_{i0}) = 0. \tag{4}$$

3. Given any $t \geq 0$, we have

$$x = \sum_{i=1}^{I} \sum_{j=0}^{m_i} \lambda_{ij} a_{ij} + t \sum_{i=1}^{I} \sum_{j=1}^{m_i} \beta_{ij}(a_{ij} - a_{i0})$$

$$= \sum_{i=1}^{I} \left[\sum_{j=1}^{m_i} (\lambda_{ij} + t\beta_{ij})\, a_{ij} + \left(\lambda_{i0} - t \sum_{j=1}^{m_i} \beta_{ij} \right) a_{i0} \right]. \tag{5}$$

Fix i. Observe that the sum of the coefficients of the terms $a_{i0}, \ldots, a_{im_i}$ in equation 5 is

$$\sum_{j=1}^{m_i} (\lambda_{ij} + t\beta_{ij}) + \lambda_{i0} - t \sum_{j=1}^{m_i} \beta_{ij} = \sum_{j=0}^{m_i} \lambda_{ij} = 1, \tag{6}$$

so the expression in equation 5 is in the form of equation 1 provided that each of the coeffients is strictly positive. For $t = 0$, all coefficients are strictly positive. $\beta_{ij} \neq 0$ for some i, j with $j \geq 1$; thus for t sufficiently large, the coefficient of a_{ij} will be either negative or will exceed 1, in which case the coefficient of some other term will be negative. Thus, there is some $t > 0$ such that at least one of the a_{ij} has a zero coefficient; let t be the smallest such value. By deleting any a_{ij} whose coefficients are zero, and renumbering if necessary, equation 5 becomes

$$x = \sum_{i=1}^{I} \sum_{j=0}^{\hat{m}_i} \hat{\lambda}_{ij} a_{ij} \tag{7}$$

with $\sum_{i=1}^{I} \hat{m}_i < \sum_{i=1}^{I} m_i$. Thus, we have an expression for x in the form of equation 1, but with a smaller value of $\sum_{i=1}^{I} m_i$. Repeat this process until we obtain an expression in the form of equation 1 with $\sum_{i=1}^{I} m_i \leq L$.

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Proof of Caratheodory's Theorem: In Lemma 1.3, take $I = 1$. Then we have $x = \sum_{j=1}^{m_1} \lambda_{1j} a_{1j}$ with $m_1 - 1 \leq L$; hence, we have $x = \sum_{j=1}^{m} \lambda_j a_j$ with $m \leq L + 1$.

Proof of the Shapley-Folkman Theorem: Because $\sum_{i=1}^{I}(m_i - 1) \leq L$, we have $m_i = 1$ except for at most L values of i. Let $a_i = \sum_{j=1}^{m_i} \lambda_{ij} a_{ij} \in \text{con } A_i$. If $m_i = 1$, $a_i = \sum_{j=1}^{1} \lambda_{ij} a_{ij} = a_{i1} \in A_i$, so equation 1 gives an expression for x in the form required.

2 Existence of Approximate Walrasian Equilibrium

The material in this section is taken from Anderson, Khan and Rashid [1] and Geller [2]. The assumptions in those papers are stated in terms of strict preference relations, $\succ$, rather than weak preference relations, $\succeq$; we will follow the same formulation here.

Theorem 2.1 *Suppose we are given a pure exchange economy, where for each $i = 1, \ldots, I$, $\succ_i$ satisfies*

1. *continuity: $\{(x, y) \in \mathbf{R}_+^L \times \mathbf{R}_+^L : x \succ_i y\}$ is relatively open in $\mathbf{R}_+^L \times \mathbf{R}_+^L$;*
2. *for each individual i, the consumption set is $\mathbf{R}_+^L$, i.e. each good is perfectly divisible, and each agent is capable of surviving on zero consumption;*

3. *acyclicity: there is no collection* $x_1, x_2, \ldots, x_m$ *such that* $x_1 \succ_i x_2 \succ_i \cdots \succ_i x_m \succ_i x_1$*;*

Then there exists $p^* \gg 0$ *and* $z_i^* \in D_i(p)$ *such that*

$$\frac{1}{I}\sum_{\ell=1}^{L} \max\left\{\left(\sum_{i=1}^{I} z_i^* - \sum_{i=1}^{I} \omega_i\right)_\ell,\ 0\right\} \leq 2\sqrt{\frac{L}{I}}\max\{\|\omega_i\|_1 : i = 1, \ldots, I\} \qquad (8)$$

where $\|x\|_1 = \sum_{\ell=1}^{L} |x_\ell|$*.*

The proof has much in common with the proof of the Debreu-Gale-Kuhn-Nikaido Lemma. One works on a compact subset of the interior of the price simplex.[1] One considers the same correspondence as in the Debreu-Gale-Kuhn-Nikaido Lemma, except that one uses the convex hull of the demand sets instead of the demand function. One finds a fixed point (p^*, x^*). Use the definition of the correspondence to show that $\left(\sum_{i=1}^{I} x_i^*\right)_\ell \leq \sqrt{\frac{L}{I}}\max\{\|\omega_i\|_1 : i = 1, \ldots, I\}$ for $\ell = 1, \ldots, L$.[2] From the definition of the correspondence, $x^* = \sum_{i=1}^{I} x_i^*$, where $x_i^* \in \text{con } D_i(p^*)$ for all $i = 1, \ldots, I$. Use the Shapley-Folkman Theorem to find y_i^* with $\sum_{i=1}^{I} y_i^* = \sum_{i=1}^{I} x_i^*$ and $y_i^* \in D_i(p^*)$ for all but L of the individuals. Let $z_i^* = y_i^*$ for all but the L exceptional individuals, and let z_i^* be an arbitrary element of $D_i(p^*)$ for the remaining individuals; this establishes a bound on the difference between $\sum_{i=1}^{I} z_i^*$ and $\sum_{i=1}^{I} x_i^*$, which proves the desired result.

The result can also be applied in the case of indivisibilities (i.e. nonconvexities in the consumption set). In that case, one obtains a bound on the excess quasidemand rather than demand. Even with nonconvex preferences, demand has closed graph, so Kakutani's Theorem applies; with indivisibilities, however, demand need not have closed graph, so one needs to consider quasidemand, which *does* have closed graph.

[1] To be more precise, it is convenient to work on $\Delta' = \{p \in \mathbf{R}_+^L : \sqrt{\frac{L}{I}} \leq p_\ell \leq 1\,(1 \leq \ell \leq L)\}$.

[2] This bound is related to the lower bound on prices in the definition of Δ'.

References

[1] Anderson, Robert M., M. Ali Khan and Salim Rashid, "Approximate Equilibria with Bounds Independent of Preferences," *Review of Economic Studies* 44(1982), 473-475.

[2] Geller, William, "An Improved Bound for Approximate Equilibria," *Review of Economic Studies* 52(1986), 307-308.

[3] Starr, Ross, "Quasi-Equilibria in Markets with Non-convex Preferences," *Econometrica* 17(1969), 25-38.